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THE
ELEMENTS OF ALGEBRA,

WITH

NUMEROUS EXERCISES

FOR *viva voce* AND WRITTEN WORK.

BY

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PREFACE.

The Elementary Algebras in common use no longer meet the requirements of Canadian Schools, and there has, therefore, arisen among mathematical teachers and students, an imperative demand for a *New Elementary Algebra* which, in matter and methods, shall more fully represent the training that our schools now give in the elements of the science. In the preparation of a work to supply this demand, at least in part, two objects have been aimed at, 1st, to give the subject some educational value for the student who is confined to a mere primary course, and 2nd, to lay a solid foundation for advanced work in algebraic analysis.

The treatment of the subject in the following pages will be found to differ materially from that usually adopted. Among the points of difference, attention may be called to the following:

(a) Algebraic Notation, instead of being passed over as a matter of minor importance, is treated with comparative fulness. A thorough knowledge of algebraic language is the foundation of all true progress in the science, and a necessary condition that mental discipline of any real value may be derived from the study of its elements. The student must learn to think in the language of Algebra.

(b) The early introduction of easy *Arithmetical Equations* and *Problems*; not, indeed, because the "Solution of Problems" is the only thing of interest to the beginner, but because the "Science of the Equation" forms a prominent feature in Algebra, and so the *idea* of the Equation can hardly be introduced too early.

(c) Valuable Matter, not given in the text books,—in Symmetry, Exact Division, Resolution into Factors, Roots and Coefficients, etc.

(d) The Methods of treatment in the case of several important topics (Highest Common Factor, Resolution into Factors, Fractions, Equations, etc.) differ from those usually followed, and, it is hoped,

will be found more helpful to both student and teacher. They are not given for the sake of mere novelty ; they have been tested in the *Class-room*, and have won the approval of many able mathematical teachers.

(*e*) *Mental Algebra ;* a minute gradation of examples—many of them being introduced for *viva voce* work. The best teachers of Arithmetic largely employ "Mental Arithmetic." "Mental Algebra" is equally valuable for the beginner.

(*f*) *Short Methods;* some attention is given to these ; they are of worth in discipline, and essential to the equipment of the student for higher work.

(*g*) The *Great Number* and *Variety* of Examples ; it is hoped that, in this important particular, a long-felt want will be supplied.

It will be observed that, by omitting some of the more difficult articles, and solving only the "(*a*)" exercises, with here and there a few of the "(*b*)" exercises—the student may first take a BEGINNER'S course, and afterwards complete the work for a more thorough course.

As the book is specially intended to be a DRILL-BOOK FOR PUPILS, the answers to the examples have not been included ; they will, however, be published in a cheap, separate form, with hints on the more difficult questions—for the use of teachers and private students who may desire to consult them.

For higher work and applications in the Theory of Divisions, Resolution of Polynomials, Higher Equations, Determinants, Series, etc., teachers and students are respectfully referred to the *University Algebra*, published by Ginn & Co., Boston.

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ELEMENTARY ALGEBRA.

CHAPTER I.

ALGEBRAIC NOTATION.

1. Symbols of Quantity.—A QUANTITY is a definite portion of any magnitude; thus, any definite number of feet, days, years, acres, dollars, etc., is a quantity. We call 6 feet, $5, 7 months, 11 acres quantities, for they are strictly defined portions of the magnitudes named.

2. To measure a quantity by a *number* we take a certain portion of it as the *unit of comparison*, and then determine *how many times* this unit is contained in the quantity to be measured. To measure *distances*, for example, we take a foot, or a yard, or a mile, etc., as the unit, and then find how many times this unit is contained in the distance to be measured. When, for example, we call a certain distance five miles, we mean that the unit, one mile, is contained five times in the given distance. So of *time*, we call a certain definite length of time five hours, and mean that it contains the unit, one hour, five times.

3. Symbols are the marks, or characters, used to denote the number of such units in any given quantity. The marks on a scoring-sheet in a game of cricket are the *symbols* used to denote the number of runs in the match. The figures 7, 12, 4, etc., are the common symbols used in Arithmetic to denote the numbers seven, twelve, four, etc. In Algebra, we use more general symbols to denote the number of times the unit of comparison is repeated, such as a, b, c, x, y, z, α, β, γ, δ, etc.

4. A complete expression of quantity by means of numbers therefore includes two things :—

1°. The **unit of measurement**; which must, of course, be of the same *kind* as the quantity to be measured.

2°. The **number of such units** in the given quantity.

5. In elementary Algebra, we consider all quantities as expressed numerically in terms of some unit, and the symbols of quantity represent only the *purely numerical parts* of such quantities. In other words the symbols denote what are called in Arithmetic *abstract* numbers.

6. An Algebraic Expression is the expression of a quantity in algebraic symbols.

7. In the statement "*James walked 12 miles*," the number of miles is actually stated, and the 12 is therefore called a **known number,** or it is said to be *explicitly assigned*.

8. In the statement "*If from five times the number of miles James walked, ten be subtracted, the remainder will be fifty*," the number of miles, though not *directly* given, may be found from the *data* to be twelve, and is therefore said to be implied in the statement, or it is called an *implicitly assigned* number, or, more commonly, an **unknown number.**

9. In the statement "*If from the double of a number six be subtracted, the result will be the same as if three had been subtracted from that number and the remainder doubled*," the number to be doubled is assigned neither *explicitly*, nor *implicitly*, since the statement is true for any number whatever. A number of this kind which has no particular value assigned, and consequently may have any value whatever, is called an *arbitrary number*. Arbitrary numbers are usually called **known numbers,** as they are often supposed to be known, though not definitely assigned.

10. Numbers explicitly assigned are represented by means of the nine digits and zero. *Unknown* numbers (numbers implicitly assigned) and *known* (arbitrary) numbers are usually represented by the letters of the alphabet. The first letters of the alphabet, as a, b, c, etc., are generally used to represent *known* numbers, while z, y, x, w, etc., commonly stand for *unknown* numbers.

11. When any letter, as x, is used in the course of a calculation, it denotes the same number throughout. We may also represent different numbers by the same letter with marks affixed; thus, instead of writing a, b, c for three different numbers, we may represent these numbers by a_1, a_2, a_3, or by the symbols a', a'', a''' (read a prime, a second, etc.).

12. In Arithmetic the figures which represent numbers are generally themselves called numbers; and similarly, in Algebra the symbols which stand for numbers or quantities, are themselves called quantities. Letter-symbols are called *literal quantities*, and figure-symbols are designated *numerical quantities;* thus in the expression $a + b + 6 + 11$, a and b are *literal*, and 6 and 11 *numerical* quantities.

The number which a letter represents is called its *value*, and, if represented *arithmetically*, its *numerical* value.

13. Word Symbols.—Certain words and phrases occur so often in algebra that it is found convenient to represent them by easily-made symbols. Chief among these are four *symbols of relation* and three pure *word symbols.* As symbols of *relation* we have,

$=$, read *equals*, *is equal to*, *will be equal to*, etc.

$>$, read *is greater than*, thus $9 > 4$.

$<$, read *is less than*, thus $4 < 9$.

:, ::, the signs of proportion, as in Arithmetic, thus $a : b :: c : d$, or $a : b = c : d$, is read a is to b as c is to d.

The pure *word symbols* are,

$\therefore$ read, *therefore*, *consequently*, *hence;*

$\because$ read, *because*, *since;* thus, $\because a = b$, and $b = c$; $\therefore a = c$, is read "Since a equals b, and b equals c; therefore a equals c."

......, the symbol of continuation, is read *continued by the same law*, thus 1, 2, 3, 4, ... means that we are to continue the numbers by the same law, 5, 6, 7, etc.; $x_1, x_2, x_3 \ldots\ldots x_n$, etc., means x_1, x_2, x_3, x_4, x_5, and so on to x_n.

EXERCISE I.

Define quantity. Explain how a quantity is measured. What are symbols? State what is necessary for the complete expression of a quantity. How is this applied in arithmetical Algebra? Give an example of explicitly assigned, implicitly assigned, and arbitrary numbers. What other names for these numbers? What symbols are used for them? Explain the terms *literal quantities*; *numerical quantities.* Write the following at full length in words:—

1. $10 > 7$. 2. $12 < 20$. 3. $b > c$. 4. $x < y$.
5. $\because \frac{9}{9} = 1, \therefore 9\frac{9}{9} = 10$. 6. If $a < 1$ and $b > 1$, $\therefore a < b$.
7. Let $a = b$ and $b > 12$, $\therefore a > 12$. 8. $0 < a < 12$. 9. $0 < 1$.
10. $\because 3 : 9 :: 4 : 12, \therefore 3 \times 12 = 4 \times 9$. 11. If $a : b :: c : d$,
$\therefore ad = bc$. 12. $\because 7 + 3 = 10, \therefore 7 = 10 - 3$.
13. $\because 9 - 2 = 7, \therefore 9 = 7 + 2$.

14. Signs of Operation.—The fundamental operations of Algebra are Addition, Subtraction, Multiplication, Division, Involution, and Evolution. A mark used to denote that one of these operations is to be performed on a quantity is called a *sign of operation.* These *signs of operation* will now be explained.

15. The sign of Addition is $+$, read *plus.* As in Arithmetic, it denotes that the quantity before which it stands, is an *addend.* Thus $a + b$, means that b is to be added to a; so that if a represents 6 and b represents 4, $a + b$ will represent $6 + 4$, which $= 10$. Hence,

The sum of any number of quantities is expressed by writing them in a row with the sign + between every two of them.

EXERCISE II.

1. If $a = 3$, $b = 5$, and $c = 8$, find the value of

   i. $a + b$. ii. $b + c$. iii. $c + a$. iv. $a + b + c$.
   v. $a + b + a$. vi. $a + b + c + b + a$.

2. If $x = 4$, $y = 7$, and $z = 10$, prove, by finding the value of each of these expressions, that

$$x + y + z = y + z + x = z + y + x = x + z + y = z + x + y.$$

3. Read the following statements :—

   i. $a + b + c = e$. ii. $b + c + c + 1 = a + e$.
   iii. $a + a + b + b = e + 2$. iv. $a + b > c$. v. $a < b + c$.

4. Prove the statements in (3) correct if $a = 3$, $b = 4$, $c = 5$, and $e = 12$.

5. Using the sign $+$, write that

   i. 3 is to be added to 7. ii. 3 is to be added to a.
   iii. y is to be added to x.
   iv. a, m, and n are to be added together.
   v. c is to be added to the sum of x and a.

16. The sign of Subtraction is $-$, read *minus*. Placed before a quantity it denotes that that quantity is a *subtrahend*. Thus $a - b$, read *a minus b*, indicates that the number represented by b is to be subtracted from the number represented by a; so that, if a represents 6 and b represents 4, $a - b$ is equivalent to $6 - 4$, which $= 2$. Hence, to subtract a quantity, say a, from another quantity, say x, *write the subtrahend after the minuend with the sign, $-$, between them*; thus, $x - a$.

EXERCISE III.

1. If $a = 25$ and $b = 7$, what is the value of

   i. $a - b$. ii. $a - b - b$. iii. $b + b + b + 7 - a$.

2. Read the following expressions :—

   i. $a + b - c$. ii. $a - b - c$. iii. $a - b + c$.

3. Find the value of those three expressions if $a = 12$, $b = 5$, and $c = 4$.

4. Write the expressions in (2) at full length in words.

5. If $a = 15$, $b = 7$, $c = 6$, and $d = 4$, prove, by finding the value of each expression, that

$$a - b - c + d = a - b + d - c = a + d - b - c =$$
$$a - c + d - b = a - c - b + d = a + d - c - b.$$

6. Using the sign $-$, write that
   - i. 3 is to be subtracted from 7.
   - ii. 3 is to be subtracted from a.
   - iii. c is to be subtracted from 10.
   - iv. y is to be subtracted from x.
   - v. x is to be subtracted from y.
   - vi. n is to be subtracted from the sum of x and m diminished by 6.

17. When an expression is made up of several parts connected by the signs $+$, $-$, each of these parts taken with the sign immediately preceding it ($+$ being understood if no written sign precedes) is called a **Term**; thus, $a + b - c + d + e$ consists of the five *terms* $+a$, $+b$, $-c$, $+d$, $+e$. A term whose sign is $+$ is called a **positive term**; one whose sign is $-$ is called a **negative term.**

An expression consisting of but one term is called a **Monomial.**

An expression consisting of more terms than one is called a **Polynomial.**

A Polynomial of two terms is called a *Binomial.* One of three terms is called a *Trinomial.* Polynomials of three or more terms are sometimes called *Multinomials.*

18. If two terms differ only in one having the sign $+$ and the other the sign $-$, they are called *complementary* terms; thus, $+b$ and $-b$ are complementary terms in the expression $a + b - b$; so $-c$ and $+c$ are complementary terms in $a-c+b+c$.

If a pair of complementary terms occur in an expression, they may be struck out without changing the value of the expression: thus, $7 + 5 - 5 = 7$, $a + b - b = a$, $a - 7 + b - 7 = a + b$, etc. So also $a = a + b - b$, etc.

EXERCISE IV.

1. Read the following :—
   - i. $a + b - c$; $x - y - 5$; $m - 7 + n$; $b + a - b + c$.
   - ii. $x = a + b$; $x - 5 = 10$; $x + 3 = n$; $a + x = x - b + a + b$.
   - iii. $\because x + 3 = 10, \therefore x = 10 - 3 = 7$; $\because x - m = a, \therefore x = a + m$.
   - iv. $\because x + 7 > 10, \therefore x > 3$: $\because y - 5 < 7, \therefore y < 12$.
   - v. $\because +b - b = 0, \therefore a = a + b - b$.

2. Write the meaning of iii and v at full length in words.

3. Write with proper signs of operation :—

   i. a minus m plus 7 minus b minus 6 plus c.
   ii. x plus b minus a minus a plus b minus 6.

4. If $a = 25$, $b = 10$, and $c = 8$, find the value of

   i. $a + b - b$; $b - c + b + a$; $b + c + a - c + a$.
   ii. $a + b - c$; $a + b + c$; $a - c + b$; $-b + a + b - c + a$.
   iii. $a - 7 - 5 - c$; $b + 10 + c - a$; $c + a - 10 - b$; $a - c + 10 - b + b$.

5. To k add l and from the result take q.

6. From a take b and then c from the remainder.

7. Express in symbols :—

   i. The difference of p and q; the sum of p, q, and r.
   ii. The number by which a exceeds c.
   iii. The number by which the sum of a and c exceeds e.
   iv. The statement that the sum of a and b is greater than c.
   v. The statement that the difference between a and b is less than c.
   vi. The number by which the sum of x, y, and z exceeds w.

EXERCISE V.

1. Express in symbols :—

   i. That x diminished by 3 is equal to four.
   ii. That x is four more than 3.
   iii. That x is equal to the sum of a and b diminished by c.
   iv. That if x, y, and z be added together, the sum will be 20.
   v. That if m be taken from the sum of x and a, the remainder will be 5 more than c.
   vi. That if m be taken from the excess of x over n, and y be then taken from the remainder, the final remainder will be zero.

2. Express the following numbers algebraically :—

   i. The number that exceeds x by y.
   ii. The number that is 5 less than the sum of x and y.
   iii. The number which is less by m than the excess of x over y.

iv. The number which if increased by 4 would be equal to c diminished by 5.

3. James has a marbles and John has 7 more; how many has John?

4. John is x years old now; how old was he 5 years ago? How old will he be 5 years hence? How old in n years hence?

5. Thomas is x years old, and his father was 35 years old when Thomas was born; how old is his father *now*? His mother is y years younger than his father; how old is his mother?

6. I poured x gallons of water into a cask, but 5 gallons leaked out. i. How many of the x gallons remained? ii. Had there been 12 gallons in the cask before I poured any in, how many gallons would remain in the cask at the end of my labour?

7. Henry had x marbles; he exchanged m of them for n of another kind. i. How many of the first kind had he left? ii. How many altogether? iii. If the number of marbles he gave in exchange was greater than the number he received in return, how many less had he after the exchange than before?

8. James and John had 25 cents each; James gave John x cents; how many cents had each then?

19. Quantities to be multiplied together are called **Factors,** and the resulting quantity is called the *Product* of these factors. Multiplication is indicated in two ways. 1°. *By a sign of operation.* 2°. *By position.*

20. The signs of Multiplication are × and ·, read *into:* thus, $3 \cdot 4$, or 3×4, is read three *into* four, and indicates the product of the factors 3 and 4. So $3 \cdot 4 \cdot 5$, or $3 \times 4 \times 5$ indicates the *continued* product of the three factors 3, 4, and 5. In like manner, $a \cdot b$, or $a \times b$, indicates the product of the factors a and b.

21. If all the factors or all but one are represented by letters, the *signs of operation*, × and ·, are generally omitted, and the factors are written in juxtaposition; this is called *indicating Multiplication* **by position.** Thus, five times a is not (usually) written $5 \cdot a$, or $5 \times a$, but simply $5a$ (read *five a*), and $\frac{2}{3}$ of the product of m and z is (generally) written $\frac{2}{3}mz$.

22. In an expression such as $7ax + 5by - 3cz - 4wv$ (read seven ax plus five by minus three cz minus four wv) *the multiplications are to be performed before the additions and subtractions.*

EXERCISE VI.

1. If x denotes a certain number, express :—
   - i. Double the number ; five times the number.
   - ii. Three times double the number ; $2a$ times the number.
   - iii. The *sum* and the *difference* of three times the number and four times the number.
   - iv. A number seven less than five times the number.
2. If $a = 12$, find the value of
   - i. $5a - 3a$; $3 \times 5a$; $1 \cdot 2 \cdot 3 \cdot 2a$; $4a - 3a$.
   - ii. $1 \cdot 2 \cdot 3a - 20$; $3a + 2a - 10$; $4a \cdot 1 \cdot 2 \cdot 3$.
3. Read $4a + 5b + 3c$; $4ab - 5ac + 3bc$; $3ax - by + z$; $5aa - 3bb - 2cc$.
4. Write at full length in words the meaning of the expressions in question 3.
5. Express :—
   - i. x dollars in cents, and y dollars in dimes.
   - ii. x dollars and y cents in cents.
   - iii. x yards, y feet, and z inches in inches.
   - iv. a cwt., b pounds, and c oz. in ounces.

23. A quantity which multiplies another quantity is called a **Coefficient** of that other quantity. A *coefficient* (literally *co-factor*) is therefore simply a multiplier (or *co*-factor) and may be either *numerical* or *literal.* Thus,

5	is the	numerical	coefficient	of	$5amx$.
$5a$	"	literal	"	"	mx.
$5am$	"	"	"	"	x.
$5x$	"	"	"	"	am.
ax	"	"	"	"	$5m$.
am	"	"	"	"	$5x$.

24. *If no numerical coefficient is written, unity is to be understood as the actual coefficient.*

EXERCISE VII.

1. Write in as many ways as possible :—

$abx,\ bxy,\ axy,\ 3ax,\ ax + by,\ a + bxy.$

2. Find the values of these expressions if $a = 2,\ b = 5,\ x = 3.$

3. Is $a + bx$ the same as $b + ax$? Why?

4. Simplify by actual multiplication :—

i. $3a \times 5x;\ 2x \times 5a;\ 3x \cdot 4;\ 2m \cdot 3x \,.\, n.$
ii. $3a \times 3 \times m \times 4y;\ 3a \times 5x + 4y \times 5b.$
iii. $a \cdot 2b - 3b \cdot 4c + 5c \,.\, a - 2a \cdot 2b \,.\, 2c.$

5. What is the coefficient of

a in $4a$; a in an; x in $4mx$; xy in $7anxy$; abc in abc?

6. Express algebraically :—

i. Seven more than the product of a and x.
ii. The product of a and x diminished by the product of m and y.
iii. Eleven more than the excess of five times a over six times b.
iv. The difference of twelve times x and four times y, increased by the product of three times x and four times y.
v. That seven more than three times x is equal to thirteen
vi. That the sum of five times x and four times y, diminished by seven, is equal to thirteen.

7. If $m = 3$ and $y = 6$, is $my = 36$? Why?

8. Write algebraically a number of two digits, a and b,

i. When b is the unit's digit. ii. When a is the unit's digit.

9. Write algebraically the six numbers of three digits each that can be expressed by the three digits $l,\ m,\ n$.

10. If a term appears without any numerical coefficient, what coefficient must be understood?

11. Simplify $3a + \frac{1}{3}b - \frac{1}{6}a - \frac{1}{2}c - \frac{1}{3}a + \frac{1}{3}c + 2\frac{1}{2}b$, when $a = 6$, $b = 12,\ c = 12.$ 47

12. James has a marbles, and John has seven less than twice as many. How many has John?

13. A man sold 12 oxen at x dollars each, and with part of the proceeds bought 18 sheep at y dollars each. How many dollars has he left of what he received for his oxen?

14. Find the price, at x cents a yard, of a bales of cloth each containing b pieces, each piece consisting of c yards; give the answer in cents.

15. A man bought n turkeys at x cents each, and sold them at y cents each, $y > x$; how much did he gain?

16. John had m marbles; he gave n of them to James, twice as many to William, and the rest to Thomas. How many did Thomas receive?

17. John's father is not three times as old as John by four years; his mother is three years younger than his father; what is his mother's age if John is x years old?

25. The sign of Division is $\div$, read **by,** and denotes that the quantity immediately following it is a divisor. Thus $a \div b$ (read a **by** b) means that a is to be divided by b: so that if a represents 12 and b represents 4, $a \div b$ represents $12 \div 4$; that is, 3.

Division is also indicated by **position,** *i.e.*, by arranging the quantities in the form of a fraction with the *dividend* for *numerator*, and the *divisor* for *denominator;* thus, $a \div b$ may be written $\frac{a}{b}$, $ax \div by$ may be written $\frac{ax}{by}$, etc.

26. In an expression such as $\frac{ax}{m} + \frac{by}{n} - \frac{cz}{q}$, *the multiplications and divisions are to be performed before the additions and subtractions*, so that in this expression the quotient of ax by m is to be increased by the quotient of by by n, and the sum diminished by the quotient of cz by q.

EXERCISE VIII.

1. If x denotes a certain number, express :—

   i. A quarter of five times the number; three quarters of the number.

   ii. The sum of $2\frac{1}{2}$ times the number and a quarter of the number.

   iii. Twelve divided by the number, and the quotient diminished by two thirds of three fifths of the number.

   iv. Twelve divided by the number and the quotient increased by three divided by the number.

   v. Seven less than seventy divided by six times the number.

2. Read the following expressions and write their meaning at full length in words :—

   i. $\frac{a}{6}$; $\frac{5a}{12}$; $\frac{3}{5}ab$; $\frac{1}{4}\cdot 5ax$; $ax \div 3$; $5a \div 3b$; $16 \div 3a$.

   ii. $\frac{a}{5}+\frac{1}{5}$; $\frac{a}{6}-2\frac{1}{2}$; $\frac{2a}{9}+\frac{4}{a}-3\frac{1}{2}$; $3\frac{1}{8}a-\frac{1}{8}\cdot\frac{a}{3}$; $3\frac{a}{8}-3\frac{8}{a}$.

3. If $a=24$, $b=5$, $c=2$, $x=3$, find the values of the expressions in i of question 2. 4, 10, 72, 96, 384 24

4. Read the following expressions and write in words :—

   i. $\frac{2ab}{3}-\frac{3bc}{4}+\frac{4ca}{5}$; $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$.

   ii. $\frac{ax}{b}+\frac{bx}{c}-\frac{cx}{a}$; $\frac{1}{aa}+\frac{2}{ab}+\frac{1}{bb}$.

   iii. $a\frac{m}{b}-c\frac{n}{a}+a\frac{b}{c}$; $a\frac{b}{c}-b\frac{c}{a}-c\frac{a}{b}$.

✓ 5. If $a=10$, $b=6$, $c=1$, $m=3$, $n=20$, and $x=y=z=30$, find the values of the six expressions in question 4.

$43\frac{1}{2}$, 38, 277, $\frac{16}{225}$, 63, 6·7 $\frac{11}{16}$

✓ 6. Express :—

   i. x cents in dollars; x inches in feet; x oz. in pounds; x feet in miles; x minutes in days; x pounds y oz. in cwts.; x yds. y ft. in miles; x ten-cent pieces in quarter-dollars.

$\frac{x}{100}$, $\frac{x}{12}$, $\frac{x}{16}$, $\frac{x}{5280}$, $\frac{x}{1440}$, $\frac{16x+y}{16}$

7. Write with algebraic signs :—

   i. Three a by x, minus four b by y, plus five c by z.
   ii. Three ax by bc, plus five by by ca, minus seven cz by ab.
   iii. Seven xy by m, plus six xy by n, minus 8.

8. Express algebraically :—

   i. Six more than the mth part of x, increased by four less than the ath part of nx. $\frac{x}{m} + 6 - \frac{nx}{a} - 4$
   ii. Ten more than the excess of the fifth part of a over the third part of b. $\frac{a}{5} - \frac{b}{3} + 10$
   iii. The number that exceeds b by one-fifth of m times x. $b + \frac{m}{5}$
   iv. The number that falls short, by twelve, of the sum of the ath part of x and the bth part of y. $\frac{x}{a} + \frac{y}{b} - 12$
   v. That six more than two-thirds of x is equal to one-third of x diminished by three. $\frac{2x}{3} + 6 = \frac{x}{3} - 3$
   vi. That five less than one-third of five times x is equal to eleven more than one-fifth of thrice x. $\frac{5x}{3} - 5 = \frac{3x}{5} + 11$

9. If $a = 2$, $b = 3$, and $c = 4$, why is not

$$a\frac{b}{c} = 2\tfrac{3}{4}, \text{ and } \frac{a}{bc} = \frac{2}{34}?$$

10. What are the numerical coefficients in

$$\frac{3ax}{5by};\ \frac{3x}{a};\ \frac{ax}{5};\ \frac{x}{a};\ \frac{1}{4}\cdot\frac{7ax}{b}?$$

11. What are the coefficients of x in the five expressions in question 10?

12. If $x = y = z = a$, find the value of

   i. $\frac{x}{y} + \frac{y}{z} + \frac{z}{x}$; $\frac{ax}{z} - \frac{az}{y} + \frac{ay}{x}$.
   ii. $\frac{x}{y} - \frac{y}{x} + \frac{y}{z} + \frac{z}{x} - \frac{x}{y}$; $\frac{1}{2}\cdot\frac{xy}{a} + \frac{1}{3}\cdot\frac{yz}{a} - \frac{1}{4}\cdot\frac{zx}{a}$.

13. What are eggs apiece at x cents a dozen?

14. Find the price of 9 ducks at x cents a pair; and of x eggs at y cents a dozen.

15. John had x dozen marbles; he gave half of them to James, half the remainder to William, and one-third of what then remained to Henry; how many did each receive?

16. James had m marbles; he gave half of them to John and a third of them to William; how many had he left?

17. Henry walked 12 miles in x hours:
How many miles an hour did he walk?
How long did he take to walk a mile?

18. Henry walked a miles in n hours:
How many miles an hour did he walk?
How long did he take to walk one mile?

19. How many posts a feet apart from centre to centre would be required for a fence along the front of a lot x feet wide, a post to be at each end of the fence?

20. John's age is three years more than a fifth of his father's age; his father is x years old; how old is John?

21. A man cut a field of wheat in x days; how much of it did he cut per day?

22. John could cut a field of wheat in x days; his father could cut the same in y days; how much of it could both together cut in a day?

23. A pipe could fill a cistern in x hours; how much of the cistern could it fill in *one* hour? In *two* hours? In *five* hours? In n hours?

24. Two pipes let water into a cistern and a man pumps it out; one of the pipes could fill the cistern in x hours, the other could fill it in y hours, and the man could empty it in z hours. If the pipes let in more water than the man can pump out in the same time, and if both pipes are opened and the man sets to work to pump, how much of the cistern would be filled in one hour? In two hours? In five hours? In n hours?

27. A Power of a quantity is the product obtained by using that quantity a certain number of times as a multiplier, starting from *unity as first multiplicand*. The operation of forming a power is called **Involution**; the quantity used as a multiplier is called the **Base** of the power; the *number of successive multiplications* is called the **Degree** of the power; and the number indicating the degree of the power is called the **Exponent**, or Index, of the power, and is written in small characters to the right and a little above the line of the base; thus:—

$1 \times a \times a$ is represented by a^2, read *a square;* here a is the *base*, 2 is the *exponent* (or *index*); the quantity itself, viz. a^2, is of the *second degree*, and is called the second *power* of a.

$1 \cdot c \cdot c \cdot c$ is represented by c^3, read *c cube;* here c is the base, 3 is the index; and the quantity, c^3, is of the *third* degree, and is called the third power of c.

In x^5, read *x to the fifth*, x is the base, 5 is the exponent; and the quantity, x^5, is of the *fifth* degree, and is called the fifth power of x.

28. Comparing powers among themselves, the second power is said to be *higher* than the first, the third higher than the second, etc.

29. The dimensions (sometimes called *degree*) of a *term* are the number of *literal* factors it contains; this is indicated by the *sum of the exponents;* thus:—

$3a^2b^2c^3$ is of *seven* $(= 2 + 2 + 3)$ dimensions.

This term, $3a^2b^2c^3$, is said also to be of *two* dimensions in a, of *two* dimensions in b, and of *three* dimensions in c.

The dimensions of a *polynomial* are determined by the dimensions of its *highest term;* thus:—

$1 + a^2 + 3abc$ is of three dimensions, because its highest term $(3abc)$ is of three dimensions.

A polynomial is said to be *homogeneous* when all its terms have the *same* dimensions; thus:—

$$x^3 + 3x^2y + 3xy^2 + y^3 \text{ is homogeneous.}$$

30. Since the exponent denotes how many multiplications by the base are to be made, the first to be performed on unity; it follows that a^1, the first power of a,

represents $1 + 1 + 1 + \ldots$ to a terms $= 1 \times a$, or simply a.

So also, $a + a + a + \ldots$ " a " $= 1 \times a \times a \ldots\ldots\ldots a^2$.

And $a^2 + a^2 + a^2 + \ldots$ " a " $= 1 \times a \times a \times a \ldots\ldots a^3$.

etc. etc.

Hence a^0, the zero power of a, denotes that *no* multiplication by a is to be made, or in other words, that the unit-multiplicand is not to be multiplied by a. Therefore $a^0 = 1$ for any value of a whatsoever.

31. In writing a power at full length as a product, it is usual to omit the unit-multiplicand, just as it is usual to omit a unit-coefficient where such occurs; thus:—

Instead of writing $x^3 = 1 \times x \times x \times x$, we write $x^3 = x \times x \times x$.

In this method of expressing the value of a power, *the index denotes the number of times the base is taken as a factor.*

32. In an expression such as $4a^2b^3 \div c^2$ (read 4 *a square b cube by c square*), the involutions are to be performed before the multiplications and divisions; thus the factors are 4, a^2, and b^3.

In an expression such as $5a^2x^4 - 4bx^3 + 3c^2x^2 \div d^2 - 7e^4$ (read *five a square x to the fourth, minus four b x cube, plus three c square x square; by d square, minus seven e to the fourth*), the order of performing the operations is,—1°, *involutions;* 2°, *multiplications*, and *divisions;* 3°, *additions* and *subtractions.*

EXERCISE IX.

1. Find the values of 1^2; 1^3; 1^4; 2^2; 3^3; 4^4.

2. Find the values of the following:—1°, for $n=1$; 2°, for $n=2$; 3°, for $n=3$; 4°, for $n=4$:—

   i. 1^n. ii. 1^n+2^n. iii. $1^n+2^n+3^n$. iv. $1^n+2^n+3^n+4^n$.

3. Find the values of

   i. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}$. ii. $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}$.
   iii. $3^3+4^3+5^3-6^2$. iv. 3^5-5^3.
   v. $3 \times 4^2 - 4 \times 3^2$. vi. $3 \cdot 2^8 - 3^2 \cdot 8$.

4. Read the following:—

   x^2; x^4: $4x^3$: $3x^4$: a^2x; $2ax^3$; $3a^3x$; $4a^3x \div b^3$; $a^4b^4 \div b^5$.

5. Write with exponents:—

   i. xxx. ii. $aaaxxxx$. iii. $4axaxax$. iv. $aabbxxxbax$.

v. $\frac{4axax}{bbb}$. vi. $\frac{2abc \cdot 2bca \cdot 2cab}{xyzyzxxzy}$. vii. $4axaxax - 2axaax$.

viii. $4amxm + 3amya$. ix. $axaxax - 4axax + 6axa - 4aaa$.

6. Read the following :

   i. $3a^2x + 2ax^2$; $3ax^2 - 2a^2x$; $a^3 - 3a^2x + 3ax^2 - x^3$.

   ii. $a^4 - 4a^2b^2 + b^4$; $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$.

7. Find the values of the expressions in 6 (i.) when $a = 4$ and $x = 1$.

8. Write with algebraic signs :—

   i. a cube, plus a square b, plus a b square, plus b cube.

   ii. Seven x to the fourth, minus five x cube y, plus three x square y square, minus five x y cube, plus seven y to the fourth.

   iii. x to the m, plus x to the m minus n y to the n, plus y to the m plus n.

9. Express algebraically :—

   i. Seven more than the square of x; seven less than the cube of x.

   ii. The excess of $3x^2$ over seven x.

   iii. The number which exceeds the cube of x by twice the square of x.

   iv. The number which is less than twice the square of x by half the cube of x.

   v. The number which is greater by the cube of x than the excess of x over the square of a.

   vi. The sum, and the difference, of twice the square of x and thrice the square of y.

   vii. That the sum of the cube of x and twice the cube of y is to be diminished by the product of x and the square of y.

   viii. That the sum of the cubes of x, y, and z is to be diminished by thrice the product of x, y, and z.

10. How many square feet in the floor of a room x feet long and x feet wide? x feet long and xy feet wide? How many square yards in each case?

11. A rectangular room is x feet long, y feet wide, and z feet in height to the ceiling ; find

    i. The area of the walls. ii. The area of the walls, ceiling, and floor. iii. The volume of the room.

12. What would be the results in the last question if $x = y = z$?

33. Like Quantities (or Terms) are such as contain the same letters, the corresponding letters in the several quantities being each raised to the *same power*: thus:—

$5a^2b^3$, $3a^2b^3$, $-7a^2b^3$ are *like* quantities; but $3a^2b$ and $3ab^2$ are **unlike** quantities, because though they contain the same letters, the letters are of different powers.

34. If an expression contains any *like* terms, these may be united and the expression is said to be *simplified*.

Thus, as in Arithmetic, 2 dozen + 3 dozen = 5 dozen; 2 times 8 + 3 times 8 = 5 times 8, etc.; so in *Algebra*, $2ab + 3ab = 5ab$; $2a^2b^3 + 3a^2b^3 = 5a^2b^3$.

Similarly, in the case of *negative* terms; $5ab - 3ab = 2ab$; $5a^2b^3 - 3a^2b^3 = 2a^2b^3$.

35. Hence, *to reduce two or more like terms to a single equivalent term:—Form the sum of the numerical coefficients of the positive terms and also of the negative terms, then take the difference of these sums, affix the literal parts and prefix to the result the sign of those terms whose numerical coefficients give the greater sum.*

Thus, in the expression $5a^2b - 7ac^2 + 3a^2b - 6ac^2 - 4a^2b - 6a^2b + 15ac^2$, the sum of the coefficients of the *positive* terms in a^2b is 8, and the sum of the coefficients of the *negative* terms is 10; the difference of these is 2, to which we affix the literal part a^2b, getting $2a^2b$; and as the sum (10) of the coefficients of the negative terms is the greater, we prefix the sign $-$, getting $-2a^2b$; similarly combining the terms in ac^2, we get $+2ac^2$, and the whole expression is *simplified* to $-2a^2b + 2ac^2$, or $2ac^2 - 2a^2b$.

36. The Reciprocal of a quantity is 1 divided by that quantity; thus the reciprocal of a is $\frac{1}{a}$, of a^2b^2 is $\frac{1}{a^2b^2}$, etc. The product of any quantity and its reciprocal is *unity*; thus, $b \times \frac{1}{b} = 1$.

Hence a *divisor* may be replaced by its fractionally-expressed reciprocal as a *multiplier*. If, for example, the product of a and b is to be divided by m, and the quotient divided by n, this may be represented by $ab \div m \div n$, or by $ab \times \frac{1}{m} \times \frac{1}{n}$, or by $\frac{ab}{m \cdot n}$.

EXERCISE X.

1. i. Define *like* and *unlike* quantities, giving examples of both.
   ii. State the rule for the addition of like quantities.
   iii. What is meant by the *dimensions* of a term? Of an *expression?* What is a homogeneous expression?

2. i. Group together the like quantities from the following,— $3x$, $5a$, $7ax$, $-3x^2$, $2x$, $-4xa$, $-3axx$, $7xx$, $5x^2$, $3ax$, $4a$, $-5x$, $8xa$, $4x$.
   ii. Simplify each group obtained in (i).

3. If $a = 4$, $b = 3$, $c = 2$, and $e = 1$, find the value of

$$a^b;\ \frac{ab}{c} + e;\ \frac{ab}{c}e;\ \frac{ab^e}{c};\ a^b + b^e + c^e;\ a^bc^e.$$

4; 6+1; 6×1; 6; 69; 128

4. Simplify $3a^2x - 4a^2x - 8a^2x + 10a^2x - 3ax^2 + 20ax^2$. $-2a^2x + 17$

5. Write in the form $a + nb$,—
   i. $a + b + b$; $+b + b + b$; $a + b + b + b + b$.
   ii. And in the form $a \times b^n$,—$a \times b \times b \times b$; $a \times b \times b$; $a \times b$; a.

6. Simplify $\frac{5}{8}x^2 - \frac{1}{3}xy + \frac{3}{16}y^2 - \frac{1}{4}x^2 + \frac{14}{15}y^2 - y^2 + \frac{1}{2}x^2 - xy + \frac{1}{5}y^2$. $\frac{7}{6}x^2 - \frac{4}{3}xy +$

7. Define the reciprocal of a quantity. What is the product of a quantity and its reciprocal? If a quantity and its reciprocal enter as factors into the same expression, how may the expression be simplified?

8. Simplify $a^3b^3c^3 \times \frac{1}{a^3} \times \frac{1}{b^3}$; $\frac{c\cancel{d} \cdot a\cancel{b} \cdot b\cancel{c} \cdot abc}{\cancel{d}x \cdot \cancel{b}x \cdot \cancel{c}x}$; $\frac{a^2 \cdot a^3}{a^5}$.

9. i. Arrange in *Descending* powers, $2 - 7x - x^2 + 5x^3 - x^4 + 3x^5$.
   ii. And in *Ascending* powers, $y^5 - y^3 - y + 1 + y^2 + y^4$.
   iii. And in Ascending powers of y, $x^4 - x^3y^3 + xy^4 - x^5y^5 - xy + y + 1$.

37. It has been seen that *involution* is the operation of forming a power by taking the same quantity several times as a factor.

Evolution is the inverse operation of separating a power (quantity) into *equal factors*. A **Root** is one of the equal factors. If the quantity is separated (or *resolved*) into *two* equal factors, each is called the **square** root; if into three equal factors each is called the **cube** root; if into four equal factors each is called the **fourth** root, etc.

38. The Root Sign is $\sqrt{}$; the *cube* (or third) root is indicated thus $\sqrt[3]{}$; the fourth root, thus $\sqrt[4]{}$; etc. When no figure is written over the root sign, the sign always indicates the square root, thus $\sqrt{16}$ indicates the square root of 16.

EXAMPLES.

i. Since $16 = 4 \times 4 = 4^2 \quad \therefore \quad \sqrt{16} = 4.$
ii. " $27 = 3 \times 3 \times 3 = 3^3 \quad \therefore \quad \sqrt[3]{27} = 3.$
iii. " $256 = 4 \times 4 \times 4 \times 4 = 4^4 \quad \therefore \quad \sqrt[4]{256} = 4.$
iv. " $x^2 = x \times x \quad \therefore \quad \sqrt{x^2} = x.$
v. " $x^3 = x \times x \times x \quad \therefore \quad \sqrt[3]{x^3} = x.$
vi. " $x^6 = x^2 \times x^2 \times x^2 \quad \therefore \quad \sqrt[3]{x^6} = x^2.$
vii. " $27x^3 = 3x \cdot 3x \cdot 3x \quad \therefore \sqrt[3]{27x^3} = 3x.$

EXERCISE XI.

MISCELLANEOUS EXAMPLES.

1. Define *addend*, *sum*, *minuend*, *subtrahend*, *involution*, *remainder*, *multiplier*, *multiplicand*, *factor*, *product*, *evolution*, *dividend*, *divisor*, *quotient*, *power*, *base*, *exponent*, *algebraic expression*, *term of an expression*, *coefficient*.

2. If $a > b$, express algebraically the sum of a and b, their difference, their product, the quotient of a by b, the power of a to b, the reciprocal of ab, and of a to the power of b.

3. In a mixed number x is the integral part, y the numerator, and z the denominator; what is the mixed number?

4. Write in as many ways as possible, by changing the order of the terms :—
   i. $a + b$; $a - b$; $a - b + c$.
   ii. Does a change in order make a change in value?
   iii. What are complementary terms? What simplification may be made in an expression having a pair of such terms? Give three examples.

5. Write in as many ways as possible, by changing the order of the factors :—

i. ab; $ca \div b$; abc; $a^2b \div c$.
ii. Does a change in order make a change in value?
iii. Express algebraically "The quotient of one quantity by another multiplied by their product equals the square of the first quantity."

6. What is the difference between an exponent and a coefficient? Which is the coefficient and which the exponent in ax^n? What is the value of this if $x = 1$, $a = 3$, $n = 5$?

7. Substitute 3 for x and find results in

$$\sqrt{12x};\ \sqrt{9x^2};\ \sqrt{91 - 9x};\ \sqrt{x^4 - 17};\ \sqrt[4]{9x^2}.$$

8. What principles are illustrated by the following:—

i. $a + b - c = a - c + b = b + a - c = -c + a + b$.
ii. $ab \div c = a \div c \times b = ba \div c = b \div c \times a$.

9. If n be a whole number, express:—

i. An even number. ii. An odd number. iii. The three consecutive numbers of which n is the middle one. iv. The three consecutive *even* numbers of which $2n$ is the middle one. v. The three consecutive *odd* numbers of which $2n + 1$ is the middle one.

10. Two numbers a and b are to be added together, the sum divided by $x + y$, and the quotient subtracted from m times k to the power of n. Express this algebraically.

11. A grocer mixed a pounds of tea worth x cents a pound, with b pounds worth y cents, and c pounds worth z cents; what is one pound of the mixture worth?

12. Two men start on a journey of c miles, the first going $a + 1$ miles an hour, and the second a miles an hour. By how much time will the first beat the second?

13. A train travels a distance of $2a$ miles, going half the distance at the rate of x miles an hour, and the other half at the rate of b miles an hour; what was the whole time taken? what was the average speed per hour?

14. Compare Involution and Evolution: what is a *root*? What is the meaning of $\sqrt[3]{x^3}$? Of a^3? a^2? a^1? a^0?

CHAPTER II.

COMPOUND EXPRESSIONS.—ADDITION.—SUBTRACTION.

39. Compound Expressions.—Any algebraic expression, however complex, represents a quantity and may be treated in any operation as a **single symbol.** If any expression is to be so treated, it is generally inclosed in *brackets*, or a line called a *vinculum* is drawn over it; thus:—

$7 + (8 - 3)$ denotes that 3 is to be subtracted from 8 and the remainder added to 7.

$7 - (8 - 3)$ denotes that 3 is to be subtracted from 8 and the remainder subtracted from 7.

$7 \cdot (8 - 3)$ or $7 \cdot \overline{8 - 3}$ means that 3 is to be subtracted from 8 and the remainder multiplied by 7. Similarly, suppose

$a + b - c$ is to be operated on as a single symbol; then

$x + (a + b - c)$ denotes that the quantity is to be added to x.
$x - (a + b - c)$ " " " " " subtracted from x.
$x(a + b - c)$ " " " " " multiplied by x.
$(a + b - c) \div x$, or
$\frac{a + b - c}{x}$ " " " " " divided by x.
$(a + b - c)^3$ " " " " " *cubed.*
$\sqrt[3]{(a + b - c)}$ " " " cube root of the quantity is to be extracted.

EXERCISE XII.

1. Find the values of

$$20 - 8 - (4 + 3);\ (20 - 8 - 4) + 3;\ (20 - 8 + 4) \div 3.$$

5 11 14

2. If $x = 40$, $y = 15$, $a = 8$, $c = 5$, find the values of $x - y + a - c$; $x - (y + a - c)$; $x - (y + a) - c$.

28; 22; 12

3. Read the expressions in the last question.

4. Distinguish between $a - c + e$ and $a - (c + e)$; also between $a - c - e$ and $a - (c - e)$.

5. Express algebraically and calculate :—

   i. 47 diminished by the excess of 22 over 15.
   ii. 10 augmented by the excess of 15 over 8, and the sum diminished by 11.

6. Express in words :—

   i. $a - x + (c - y)$; $a - (x + c) - y$; $a - (x + c - y)$.

7. If $a = 60$, $b = 10$, $c = 6$, and $d = 2$, find the values of

   i. $a(b - c + d)$; $ab - c + d$; $a(b - c) + d$; $(a - b)(c - d)$.
   ii. $(a - b)c - d$; $a - b(c - d)$; $a - bc - d$; $(a - b + c)d$.
   iii. $2(a - b)c - d$; $(2a - b)(c - d)$; $4(2a - 3b)c - 100d$.
   iv. $\sqrt{\{3a - 10(b - c)d\}}$; $\sqrt[3]{\{(a - 5b)^2 - 9(c - d)\}}$; $\sqrt[4]{(6b^2 + \frac{5}{12}a)}$.

8. Express algebraically that

   i. The sum of x and a is to be diminished by c.
   ii. Six times x is to be diminished by the sum of a and c.
   iii. Thrice the excess of x over a is to be diminished by four times the sum of y and c, and 4 times c added to the remainder.
   iv. That from the square root of the excess of x square over a square, is to be subtracted the cube root of the excess of x square over a square.

40. An expression having *a part enclosed in brackets* may itself be enclosed in brackets to form part of a *longer expression;* this again may be enclosed in brackets to form part of a still longer expression; and so on to any extent. When several pairs of brackets are thus used it is usual to make each pair different from the others in size or shape. Thus,

1°. If it were required to multiply $a + b$ into the sum of $a(x + y)$ and $b(x - y)$, the result would by expressed thus,

$$(a + b)\{a(x + y) + b(x - y)\}.$$

2°. $10a - [5b - \{4c + 2(3b - a)\}]$,—read, "10 a minus bracket $5b$ minus bracket $4c$ plus two bracket $3b$ minus a, close all brackets,"—

denotes 1°, that a is to be subtracted from $3b$, 2°, the remainder to be doubled, 3°, the product to be added to $4c$, 4°, the sum to be subtracted from $5b$, 5°, the remainder to be subtracted from $10a$. Hence if $a = 15$, $b = 6$, and $c = 1$, we shall have

$$\begin{aligned}
&10a - [5b - \{4c + 2(3b - a)\}] \\
&= 150 - [30 - \{4c + 2(18 - 15)\}] \\
&= 150 - [30 - \{4 + 2 \times 3\}] \\
&= 150 - [30 - \{4 + 6\}] \\
&= 150 - [30 - 10] \\
&= 150 - 20 \\
&= 130.
\end{aligned}$$

EXERCISE XIII.

1. Multiply the difference of the quantities a and b by d and subtract the product from c.

2. Write down the expression which indicates that b is to be subtracted from a, that remainder from c, that remainder from d, and that remainder from c.

3. Multiply the result in question (1) by $x + y$.

4. Express algebraically that the square of the difference between a and b is to be added to $2ab$, and the sum subtracted from the square of the sum of a and b.

5. If $b = 3$ and $a = 2$, find the values of

   i. $3\{b + \frac{1}{2}(a - b)\}$.
   ii. $(3b^2 - 2a^2)\{a - 4(b + \overline{a - 2b})\}$.
   iii. $a^2 + b^2 - \{b(a + b) - a(a - b)\}$.
   iv. $7a - [5b - \{4a - (3a - 2b)\}]$.

6. Write the expression which means that 1°, $5x$ is to be taken from $6x$, 2°, the remainder subtracted from $4x$, this remainder to be taken from $3x$, and the result from $2x$.

7. Express in words :—

   i. $8(b - c) - \{-(a - b) - 3(c - \overline{b - a})\}$.
   ii. $-3[-2\{-4(-a)\}] \div 5[-2\{-2(-a)\}]$.

41. Addition.—*The Algebraic Sum* of any number of quantities may be expressed by writing them in a row with their proper signs. Thus the sum of the quantities $7ax$, $5by$, $-3am$, $7bc$, and $-3abc$, is $7ax + 5by - 3am + 7bc - 3abc$.

42. If some of the terms in the expressions to be added are *like* quantities *these may be combined by adding the coefficients and affixing the literal part to the sum.* (Art. 35.)

Ex. 1. $3a - 4a + 5a + 7a - 6a$; here the sum of coefficients of + terms is $(3 + 5 + 7) = 15$.

Sum of coefficients of the negative terms is $(4 + 6) = 10$;

$$\therefore (15 - 10)\,a = 5a, \text{ is the result.}$$

For *convenience* we may set the quantities down in rows with the *like* terms under one another.

Ex. 2. Add $3a^2x - 3by^2 + 2x$, $-3x + 4by^2 + 4a^2x$, $-6by^2 + 5x - 2a^2x$, and $5a^2x + by^2 + x$. Arrange thus:

$$\begin{array}{rrrr}
 & 3a^2x & -\ 3by^2 & +\ 2x \\
 & 4a^2x & +\ 4by^2 & -\ 3x \\
 & -\ 2a^2x & -\ 6by^2 & +\ 5x \\
 & 5a^2x & +\ \ by^2 & +\ \ x \\
\hline
\text{Sum} = & 10a^2x & -\ 4by^2 & +\ 5x
\end{array}$$

N. B.—Each new quantity is written to the right in a new column.

EXERCISE XIV.

(*a*)

Add the following quantities:—

1. 5 and 7; 5 and -7; $-7a$ and $+9a$; $-9a$ and $+4a$; x and $-\frac{2}{3}x$; $3abc$ and $-5abc$.

2. $\frac{3}{4}xy^2$ and $-2xy^2$; $15a^2bc$ and $-10a^2bc$.

3. $4(a^2 + b^2)$ and $3(a^2 + b^2)$; $7(a^2 - b^2)$ and $-8(a^2 - b^2)$; $7\sqrt{x}$ and $-9\sqrt{x}$.

4. $7(a^2 + b^2 - c^2)$ and $3(a^2 + b^2 - c^2)$.

5. $-8(x^2 + y^2 + z^2)$ and $-4(x^2 + y^2 + z^2)$.

6. $x^2 + y^2$ and $x^2 - y^2$; $2a - 2b$ and $-2a$.

7. $3 \cdot \frac{a}{x}$ and $4 \cdot \frac{a}{x}$; $-7 \cdot \frac{a+b}{a-b}$ and $-3 \cdot \frac{a+b}{a-b}$.

8. $x + y + z$ and $x - y - z$; $3ab$, $-7ab$, and $8ab$.

(*b*)

1. $6x^2 - 3x^3y - y^2$, $-13x^3y + 2xy^3$, $4x^3y - xy^2 - 3y^3$, and $8xy^2$.
2. $2(m + n) + 3(a + b)$, $(a + b) - 3(m + n)$, $(a + b) - (m + n)$, and $-6(m + n)$.
3. $7a^3 - 2a^2 + 3ax$, $-a^3 - a^2 - ax$, $-6a^3 + 3a^2 - 2ax$.
4. $2(m + n)^2 + x$, $3(m + n)^2 - y$, $4(m+n)^2 - 2x$, $-(m+n)^2 + x$.
5. $x^3 - 4x^2y + 6xy^2$, $2x^2y - 3xy^2 + 2y^2$, $y^3 + 3x^2y + 4xy^2$.

   $x^3 + x^2y + 7xy^2 + 2y^2 + y^3$
6. $\frac{1}{2}a^3 - 2a^2b - \frac{3}{2}b^3$, $\frac{3}{2}a^2 - \frac{3}{4}ab^2 + 2b^3$, $-\frac{3}{2}a^3 + ab^2 + \frac{1}{2}b^3$.
7. $\frac{x}{y} - 2\frac{m}{n}$, $2\frac{x}{y} - 4\frac{m}{n}$, $3\frac{x}{y} - 3\frac{m}{n}$, $4\frac{x}{y} - 4\frac{m}{n}$. $10\frac{x}{y} - 13\frac{m}{n}$

43. When several terms contain a common *literal* part multiplied by different coefficients, *the coefficients may be collected and their aggregate prefixed to the common literal part:* thus:—

as $\quad 7x + 8x - 4x = (7 + 8 - 4)x$,

so $\quad ax + bx - cx = (a + b - c)x$. In like manner,

$$mnx + 2by + pqx - 4by = (mn + pq)x + (2b - 4b)y$$
$$= (mn + pq)x - 2by.$$

EXERCISE XV.

Collect the coefficients in the following:—

1. $3x - 2y + 6bx - 4y + 7ax + m + n$.
2. $ax - dy + mx + ny + x + y$.
3. $\frac{2}{3}ay - 2x + \frac{3}{4}by + 6ax - 3bx + y$.
4. $2dx + 3ey + 4fz - 2fx - 3dy + 4cz$.

5. $\frac{1}{2}a^2y - 4x + c^2y - \frac{2}{3}a^3y - 7y + y + (a + b)x.$

$(a+b-4)x + (\frac{1}{2}a^2 + c^2 - \frac{2}{3}a^3 - 6)y$

6. $ax - by + (a - b)x - (a + b)y + (a + b)x - (b - a)y.$ 3 [illegible] $- 3$ [illegible]

7. $5m\sqrt{y} - 6x + 4\sqrt{y} - 3\sqrt{x} - y + \sqrt{y} + ax + by.$

$(a-6)x + (5m + 5)\sqrt{y} + (b-1)y - 3\sqrt{x}$

8. $(a - b)x^2 + (b - c)y^2 + (c - a)z^2 + (b - c)x^2 + (c - a)y^2$
$+ (a - b)z^2 + ax + by + cz.$ $(a-c)x^2 + (b-a)y^2 +$ [illegible] $z^2 +$ [illegible]

9. $x^n - ax^n - 3bx^n - 9ax^n + 7by^n + (a + b - c)x^n + 10ay^n + y^n - 3abcy^n.$

$(1 - 9a - 2b - c)x^n + (1 + 7b + 10a -$ [illegible]

44. Subtraction.—*The Algebraic Difference* of two quantities is found by *changing the signs of all the terms of the subtrahend*, or conceiving them to be changed, *and combining them*, as in addition, *with the terms of the minuend.*

Ex. 1 . Suppose that 7, which is 3 less than 10, or (10 — 3), is to be subtracted from 15, the work is indicated thus : 15—(10—3). Subtract 10, the result is 5, which is too small by 3, since not 10, but *three less than ten*, is to be subtracted. The true remainder is therefore 5 + 3 ; thus :—

15	*Minuend.*
$10 - 3$	*Subtrahend.*
$5 + 3$	*Remainder.*

Ex. 2 . Similarly if a, b, c represent any quantities whatever, and $b - c$ is to be subtracted from a ; then, we have $a - (b - c)$: subtracting b we have $a - b$, which is too small by c, since not b, but c units *less than* b is to be taken from a. ∴ The true remainder is $a - b + c$; or

a	*Minuend.*
$b - c$	*Subtrahend.*
$a - b + c$	*Remainder.*

Ex. 3 . From $5x^2 - 4xy + 5y^2$ take $2x^2 - 6xy + 7y^2$.

$5x^2 - 4xy + 5y^2$	*Minuend.*
$-2x^2 + 6xy - 7y^2$	*Subtrahend*, with signs changed.
$3x^2 + 2xy - 2y^2$	*Remainder*, found as in Addition.

Ex. 4. From $ax^2 - by^2 - 2cz^2$

take $px^2 - qy^2 + rz^2$

$$ax^2 - px^2 + qy^2 - by^2 - rz^2 - 2cz^2.$$

Or, collecting, as in addition, the coefficients of the like powers of x, y, and z, we have

$$(a - p)x^2 + (q - b)y^2 - (r + 2c)z^2.$$

EXERCISE XVI.

(a)

1. Subtract 3 from 12; -3 from 12; -3 from -12.
2. Subtract $3x$ from $12x$; $-3x$ from $12x$; $-3x$ from $-12x$.
3. Subtract $-3x^2y$ from $7x^2y$; from $-17x^2y$; from xy^2.
4. Subtract $6\sqrt{x}$ from $12\sqrt{x}$; $-6\sqrt{x}$ from $-9\sqrt{x}$.
5. Subtract 0 from $5x$; $5x$ from 0; $x - y$ from 0.
6. Subtract $-5(a + b)$ from $4(a + b)$; from $-8(a + b)$.
7. From $2b$ take $a + b - c$; and also $\frac{1}{2}a - \frac{1}{2}b - \frac{1}{2}c$.
8. From $a - b$ take i. $a^2 - b^2 - c^2$, and ii. $-7(a - b)$.
9. What must be taken from $-x - y$ to obtain $x + y$?
10. A boy bought x oranges at 5 cents a piece; he sold y of them for 7 cents each, and the rest at 8 cents each. What did his sales amount to?

(b)

$-9x^4 - 2x^3y + 6x^2y^2 - 18xy^3 + 2$ [illegible] $+$ [illegible]

1. From $x^4 + 4x^3y + 6x^2y^2 - 4xy^3 + y^4$
   take $-8x^3y + 14xy^3 - 20y^4 - 20 + 10x^4$.
2. From $p^2 + 3q^2 - 11r^2 + 18pq - 8yz^2 - 1$
   take $20q^2 + 11r^2 + pq - 100$.

$p^2 - 17q^2 - 22r^2 + 17pq -$ [illegible] $z^2 +$ [illegible]

3. From $5(x - y) + 7(x - z) - 9(z - x)$ $-3(x - y) + 34(x - z)$
   take $8(x - y) - 7(x - z) + 11(z - x)$.
4. From $12(a - b) - 3(a + b) + 7a - 2b$
   take $-7(a - b) + 5(a + b) - 7a - 3b$.

$19(a - b) - 8(a + b) + 14a + b$

$12\frac{x}{y} - 11\frac{y}{x} - 8\frac{z}{x} - 8\frac{a}{b} - 6\frac{y}{z}$

5. From $7\frac{x}{y} - 11\frac{y}{x} - 15\frac{z}{x}$ take $-5\frac{x}{y} + 6\frac{y}{z} - 7\frac{z}{x} + 8\frac{a}{b}$.

6. From $\frac{1}{2}x^2 - \frac{1}{3}xy - \frac{3}{2}y^2$ take $-\frac{3}{2}x^2 + xy - y^2$. $2x^2 - \frac{4}{3}xy - \frac{1}{2}y^2$

7. From $\frac{1}{8}a^3 - 2ax^2 - \frac{1}{3}a^2x$ take $\frac{1}{3}a^2x + \frac{1}{4}a^3 - \frac{1}{2}ax^2$. $-\frac{1}{8}a^3 - \frac{2}{3}a^2x - \frac{3}{2}ax^2$

8. From $6x^3 - 7x^2y + 4xy^2 - 2y^3 - 5x^2 + xy - 4y^2 - 2$
take $8x^3 - 7x^2y + xy^2 - y^3 + 9x^2 - xy + 6y^2 - 4$.
$-2x^3 + 3xy^2 - y^3 - 14x^2 + 2xy - 10y^2 + 2$

9. From $a^2b^2 - a^2bc - 8ab^2c - a^2c^2 + abc^2 - 6b^2c^2$
take $2a^2bc - 5ab^2c + 2abc^2 - 5b^2c^2$.
$a^2b^2 - 3a^2bc - 3ab^2c - a^2c^2 - abc^2 - b^2c^2$

10. What number must be taken from $a^4 + 2a^2b^2 + b^4$ to give the remainder $4a^2b^2$? $a^4 - 2a^2b^2 + b^4$

11. From $a^2x^2 + b^2y^2 + c^2z^2$ take $b^2x^2 - c^2y^2 + a^2z^2$. $(a^2-b^2)x^2 + (b^2+c^2)y^2 +$

12. $10a^m - 15b^n - c^p + 5d^q$ take $-9a^m + 2b^n + c^p - 5d^q$.
$19a^m - 17b^n - 2c^p + 10d^q$

13. From $ax^3 - bx^2 + x - 7$ take $px^3 - qx^2 + rx - 8$.
$(a-b)x^3 + (q-b)x^2 + (1-r)x + 1$

14. What number must be taken from $2a^3 - 6a^2b + 6ab^2 - 2b^3$ to leave $a^3 - 7a^2b - 3b^2$? $a^3 + a^2b + 6ab^2 - 2b^3 + 3b^2$

15. From what number must $3y + \frac{11}{4}a - \frac{2}{3}x$ be subtracted to give $-\frac{5}{2}y - \frac{59}{12}a - \frac{1}{12}x$? $\frac{1}{2}y - \frac{13}{6}a - \frac{3}{4}x$

16. From $\frac{6}{7}(xyz - bx + cy)^2 - 1\frac{1}{10}(z - y + xa) - 2\frac{1}{2}(ax + by^2 - c^2z)$
take $2\frac{1}{7}(xyz - bx + cy)^2 + 3\frac{1}{5}(z - y + ax) - 2\frac{1}{2}(ax - c^2z + by^2)$.
$-\frac{2}{7}(xyz - bx + cy)^2 - 4\frac{3}{10}(z - y + ax)$

17. Add together the eight following expressions; from the sum subtract the first expression; from the remainder subtract the second expression; and so on, till nothing remains:—

i. $2\frac{1}{2}a - 3\frac{1}{4}b + 6\frac{3}{4}c - 5\frac{1}{3}d - 4\frac{1}{4}e$; $37\frac{3}{4}a - 12\frac{13}{24}b - 4\frac{17}{24}c -$ [illegible]
ii. $1\frac{1}{3}a + 2\frac{1}{3}b - 5\frac{1}{2}c + 4\frac{3}{4}d - 3\frac{1}{3}e$; $35\frac{1}{4}a - 9\frac{5}{24}b - 1$ [illegible] $-1\frac{5}{6}$
iii. $2\frac{1}{4}a + 1\frac{1}{4}b - 2\frac{3}{4}c - 1\frac{1}{3}d + 1\frac{1}{4}e$; $33\frac{11}{12}a - 11\frac{5}{6}b - 5\frac{23}{24}c - 6\frac{7}{12}d -$ [illegible]
iv. $2\frac{1}{2}a - 4\frac{3}{4}b - 6\frac{1}{3}c - 7\frac{1}{8}d - 5\frac{1}{6}e$; $31\frac{2}{3}a - 12\frac{7}{8}b - 3\frac{5}{24}c - 5$ [illegible] -22
v. $6\frac{3}{4}a - 7\frac{3}{4}b + 7\frac{3}{8}c + 6\frac{1}{3}d - 4\frac{1}{4}e$; $29\frac{1}{6}a - 8\frac{1}{8}b + 3\frac{1}{6}c + 1\frac{1}{8}d - 1$
vi. $5\frac{3}{4}a - 3\frac{1}{4}b - 2\frac{1}{3}c - 3\frac{3}{8}d - 5\frac{1}{4}e$; $22\frac{5}{12}a - \frac{3}{8}b - 4c - 4$ [illegible] -12
vii. $7\frac{1}{3}a - 5\frac{5}{8}b - 6\frac{2}{3}c - 7\frac{3}{4}d - 3\frac{1}{6}e$; $16\frac{2}{3}a + 2\frac{7}{8}b - 1\frac{4}{7}c - 1\frac{1}{2}d - 7$
viii. $9\frac{1}{3}a + 8\frac{1}{2}b + 4\frac{3}{4}c + 6\frac{2}{3}d - 4\frac{1}{3}e$. $9\frac{1}{3}a + 8\frac{1}{2}b + 4\frac{3}{4}c + 6\frac{2}{3}d -$

18. What do the following expressions become, when $x = a$?— 0

i. $(x^2+a^2)^2 - 4ax(x^2 - ax + a^2)$; $\sqrt{(x^2 - ax + a^2)}$; $\sqrt[3]{(x^3 + ax^2 + a^3)}$. 0, a, $\sqrt[3]{3}$

ii. $(x^2 + ax + a^2)^3 - (x^2 - ax + a^2)^3$; $\sqrt[3]{\{a^3 + x^3 + 3ax(a + x)\}}$.

$26a^6$ $2a$

CHAPTER III.

EASY EQUATIONS AND PROBLEMS.—REMOVAL OF BRACKETS.

45. We shall now give a few easy problems showing (i.) how Algebra is applied in their solution, and (ii.) how certain algebraic expressions may be read as problems. In the arithmetical solutions the signs +, −, are used. In the algebraic solutions, x *will represent the number sought*, the arithmetical *equations* are placed on the left hand, and the algebraic, on the right hand.

Ex. 1 . If 6 be taken from double a certain number the remainder will be 4 ; find the No.

ARITHMETICAL EQUATIONS.	ALGEBRAIC EQUATIONS.
2 times No. − 6 is 4	$2x - 6 = 4$
∴ 2 " " is $4 + 6 = 10$	$\therefore 2x = 4 + 6 = 10$
∴ " is $10 \div 2 = 5$	$\therefore x = 10 \div 2 = 5.$

Ex. 2 . If $\frac{3}{4}$ of a number be increased by 6 the sum is 15 ; find the number.

$\frac{3}{4}$ of No. + 6 is 15	$\frac{3}{4}x + 6 = 15$
∴ $\frac{3}{4}$ " " is $15 - 6 = 9$	$\therefore \frac{3}{4}x = 15 - 6 = 9$
∴ $\frac{1}{4}$ " " is $9 \div 3 = 3$	$\therefore \frac{1}{4}x = 9 \div 3 = 3$
∴ No. is $3 \times 4 = 12$	$\therefore x = 3 \times 4 = 12.$

Ex. 3 . A post sunk in a pond is $\frac{1}{4}$ of its length in the water, $\frac{1}{3}$ of its length in the mud, and stands 10 ft. out of water : find the entire length of the post.

length = $\frac{1}{4}$ length + $\frac{1}{3}$ length + 10 ft.	$x = \frac{1}{4}x + \frac{1}{3}x + 10$
= $\frac{7}{12}$ length + 10 ft.	$= \frac{7}{12}x + 10.$
∴ length − $\frac{7}{12}$ length = 10 ft.	$\therefore x - \frac{7}{12}x = 10.$
Or $\frac{5}{12}$ length = 10 ft.	Or $\frac{5}{12}x = 10$
∴ $\frac{1}{12}$ length = $10 \div 5 = 2$ ft.	$\therefore \frac{1}{12}x = 10 \div 5 = 2.$
∴ length = $2 \times 12 = 24$ ft.	$\therefore x = 2 \times 12 = 24,$

Ex. 4. A purse and the money it contains are worth $24, and the money is worth 11 times the value of the purse. What is the purse worth?

Value of purse + 11 times value of purse is $24;	$x + 11x = 24$;
Or 12 times val. of purse = $24;	Or $12x = 24$;
∴ " " " = $24÷12=$2.	$\therefore x = 24 \div 12 = 2$.

Ex. 5. Divide $95 among A, B, and C, so that A shall have $10 less than B, and C $15 more than B.

We have:—B's share:	Let x denote B's share;
and A's share is B's " − $10,	then $x - 10 =$ A's,
" C's " " B's " + $15.	and $x + 15 =$ C's.
∴ B's + B's − 10 + B's + 15 = $95.	$\therefore x+x-10+x+15=95$.
Or 3 times B's share + $5 = $95,	Or $3x + 5=95$.
and 3 times B's = $95 − $5 = $90;	and $3x=95-5=90$;
∴ B's share = $90 ÷ 3 = $30, etc.	$\therefore x = 90 \div 3=30$, etc.

Ex. 6. Give the expression $4x - 2 = 3x + 3$ as a word problem and solve it.

"If 4 times a certain number be diminished by 2, the remainder will be equal to 3 more than 3 times the number." Find the number.

4 times No. − 2 is 3 times No. + 3;	$4x - 2 = 3x + 3$;
∴ 4 times No. − 3 times No. − 2 is 3.	$\therefore 4x - 3x - 2 = 3$.
Or No. − 2 is 3;	Or $x - 2 = 3$;
∴ No. = 3 + 2 = 5.	$\therefore x = 3 + 2 = 5$.

Ex. 7. $\frac{x}{2} - \frac{x}{4} = \frac{x}{5} + 3$. "If a quarter of a certain number be subtracted from half of it, the remainder will be 3 more than one-fifth of the number."

$\frac{1}{2}$ the no. − $\frac{1}{4}$ the no. is $\frac{1}{5}$ the no. + 3.	$\frac{x}{2} - \frac{x}{4} = \frac{x}{5} + 3$.
Or $\frac{1}{4}$ the no. is $\frac{1}{5}$ the no. + 3;	Or $\frac{1}{4}x = \frac{1}{5}x + 3$;
∴ $(\frac{1}{4} - \frac{1}{5})$ of the no. is 3.	∴ $(\frac{1}{4} - \frac{1}{5})x = 3$.
Or $\frac{1}{20}$ of the no. is 3;	Or $\frac{1}{20}x = 3$;
∴ The *no.* is 3 × 20 = 60.	∴ $x = 3 \times 20 = 60$.

N. B.—Such algebraic expressions as these, in which the number is not given *directly*, but may be found by changing the forms of the expressions, are called **Equations.** (See Art. 8.)

EXERCISE XVII.

(*a*)

Find x in the following :—

1. $x-3=0$; $x-12=0$; $x-2\frac{1}{2}=0$; $x+4=0$; $x-a=0$; $x+a=0$.

2. $x-3=4$; $x-12=-4$; $x-4=5\frac{1}{2}$; $x-a=b$; $x+b=a$.

3. $7=x-4$; $4=x-6$; $ba=x+2$; $3a=x-3$; $4=x+3a$.

4. $\frac{3}{4}x=16$; $\frac{3}{4}x=\frac{1}{2}x+8$; $6x=15$; $7x=3x+20$; $ax=b$.
$\frac{64}{3}$ 32 $2\frac{1}{2}$ $x = 5$ $\frac{b}{a}$

5. $10x=\frac{8}{9}$; $4x=\frac{3}{2}-x$; $7x=0$; $x-a-b=0$; $ax=0$.
$\frac{4}{45}$ $x=\frac{3}{10}$ $x=0$ $x=a+b$ $x=0$

6. $ax=\frac{b}{c}$; $bx=\frac{b^2}{a^2}$; $ax+bx=1$; $ax-bx=a-b$.
$\frac{b}{ac}$ $\frac{b}{a^2}$ $\frac{1}{a+b}$ $x=1$

7. $5x=6+2x$; $10x+4=9x+40$; $4x-20=8-3x$.
2 36 4

8. $2-4x=7+x-5$; $x-a=b+c$; $x+a-b=a+b$.
0 $a+b+c$ $2b$

9. $\frac{1}{2}x=3$; $\frac{2}{3}x=6$: $\frac{3}{4}x=9$; $\frac{4}{5}x=16$; $\frac{1}{a}x=b$; $\frac{2}{a}x=c$.
6 9 12 20 ab $\frac{ac}{2}$

10. $\frac{5}{6}x=\frac{3}{4}$; $\frac{a}{b}x=c$; $x+a+b=0$; $\frac{a}{b}x=\frac{b}{a}$; $5x=1$; $.01x=1$.
$\frac{9}{10}$ $\frac{bc}{a}$ $-a-b$ $\frac{b^2}{a^2}$ $\frac{1}{5}$ 100

11. $.7x=.07$; $.01x=1.3$; $.001x=4$; $\frac{x}{a+b}=c$; $\frac{x}{a-b}=3$.
$\frac{1}{10}$ 130 4000 $c(a+b)$ $3(a-b)$

12. $\frac{x}{2}-\frac{x}{3}=1$; $\frac{x}{3}+\frac{x}{4}=\frac{1}{2}$; $\frac{x}{4}+\frac{a}{2}=9$; $\frac{x}{2}-3=\frac{x}{4}+3$.
6 $\frac{6}{7}$ $36-2a$ 24

13. $\frac{1}{x}=\frac{1}{9}$; $\frac{2}{x}=\frac{2}{7}$: $\frac{1}{x}=\frac{1}{a}$; $\frac{2}{x}=\frac{4}{ab}$; $\frac{2}{x}=\frac{4}{7}$; $\frac{a}{x}=\frac{b}{c}$
9 7 a $\frac{ab}{2}$ $\frac{7}{2}$ $\frac{ac}{b}$

14. $\frac{1}{ax}=\frac{1}{b}$; $\frac{a}{x}=abc$; $\frac{1+b}{x}=a+b$; $\frac{2}{x}-\frac{1}{4}=\frac{1}{x}+\frac{1}{2}$.
$x = \frac{b}{a}$ $\frac{1}{bc}$ $\frac{1+b}{a+b}$ $x = \frac{4}{3}$

(*b*)

1. Two persons have $24 between them; the first has three times as much as the second; how much has each? 6, 18

2. Divide $70 between A and B, giving B three-fourths of what A receives. 30, 40

3. John has 40 cents less than three times what James has; how much has each when both have $1? 35, 65

4. Three bank notes are worth $16; the first is worth $2 more than the second, and the third, $8 more than the second; find the value of each note. 4, 2, 10

5. Three numbers amount to 60; the first is one-fifth of the third and two-thirds of the second; find all the numbers. 8, 40, 1

6. A man had a sum of money at interest for 3 years at 8% per year; he spent $30 of the interest and had $90 left; find the sum at interest. $500

7. Divide $420 among A, B, and C, so that B shall have $17 more than A, and C, $26 more than B. 120 137 163

8. A certain number increased by its half, its third, its fourth, and its fifth part, amounts to 274; find the number. 120

9. John can copy a dictation exercise in a quarter of an hour; with B's help he does it in 10 minutes; how long would it take B to copy it? 30 mins

10. A youth spends $\frac{1}{3}$ of his salary for board, $\frac{1}{5}$ for clothes and books, and $\frac{1}{10}$ in other expenses, and has $308 left; find his income. $840

11. The greater of two numbers is equal to six times the smaller, and their sum is 49; find the numbers. 7, 42

12. A, B, and C rent a grazing farm for $112. A puts in 8 cattle, B, 9, and C, 11; find each man's share of the rent. 32, 36, 44

N. B.—Let x = rent due for each head of cattle.

13. In an orchard $\frac{1}{3}$ of the trees are of the same kind, $\frac{1}{4}$, of another kind, $\frac{1}{8}$, of a third, $\frac{1}{12}$, of a fourth, and there are 5 odd trees besides; how many trees of each kind are there? 24

N. B.—Let $24x =$ the whole number of trees.

8, 6, 3, 2

✓ 14. Two persons gained \$375 in a partnership, to which one contributed \$500, and the other, \$850; divide the profit between them. \$138 $\frac{8}{9}$ \$236 $\frac{1}{9}$

N. B.—Let $x =$ profit due to \$1 capital.

✓ 15. After paying an income tax of 7d. on the £, I have £1165 left; find my gross income. £1200

N. B.—Let $240x =$ gross income.

✓ 16. In a certain business, \$2000 produces \$50 profit in 3 mos.; how long would it take for \$3000 to produce \$175 profit?

N. B.—Let $x =$ no. of months. 7 months

✓ 17. Two-fifths of A's money is equal to B's, and seven-ninths of B's is equal to C's. In all they have \$770; what has each?

N. B.—Let $45x =$ A's money; $\therefore 18x =$ B's, etc.

450, 180, 140

✓ 18. A hare is 50 leaps ahead of a hound, and takes 4 leaps to his 3; but two of the hound's leaps are as long as three of the hare's; how many leaps must the dog take to catch the hare? 300

N. B.—Let $2x =$ length of hare's leap, and $\therefore 3x =$ length of hound's, etc.

✓ 19. A gamester lost $\frac{1}{5}$ of his money and then won \$10; next he lost $\frac{1}{3}$ of the remainder and won \$3; he now had \$63 left; how much had he at first? \$100

N. B.—Let $15x =$ the sum; $\therefore 12x + 10 =$ 1st rem., etc.

✓ 20. Find a number such that if $\frac{3}{8}$ of it be subtracted from 20, and $\frac{5}{11}$ of the remainder from $\frac{1}{4}$ of the original number, then 12 times the second remainder shall equal half the original number. 24

N. B.—Let $88x =$ the required no.; $\therefore 20 - 33x =$ etc.

46. Removal of Brackets.—An *aggregate* (*i. e.*, a quantity in brackets) is to be operated on as a single quantity (Art. 39), the operation to be performed being indicated by the sign immediately before the bracket.

1°. If the brackets are preceded by the sign +, the *brackets may be removed and the terms added without change of sign.*

2°. If the sign — precedes, then *the sign of every term within the bracket must be changed* (Art. 44) *on removal of the brackets.*

Ex. 1 . $21 + (7 - 3 + 4 - 2) = 21 + 7 - 3 + 4 - 2 = 27.$

2 . $21 - (7 - 3 + 4 - 2) = 21 - 7 + 3 - 4 + 2 = 19.$

3 . $a + (b - c - d + e) = a + b - c - d + e.$

4 . $a - (b - c - d + e) = a - b + c + d - e.$

EXERCISE XVIII.

Remove brackets and collect terms when possible in the following cases :—

1. $a^2 + (b^2 - c^2 - d^2)$; $a^2 - (b^2 - c^2 - d^2)$. $a^2 + b^2 - c^2 - d^2$ $a^2 - b^2 + c^2 + d$
2. $a^2 - (b^2 - c^2) - (b^2 - c^2 + a^2) - (c^2 + b^2 - a^2)$. $a^2 - 3b^2 + c^2$
3. $-(3m + 2n) - (3m - 2n) + 9m - (m + n - 6)$. $2m - n + 6$
4. $2x - (3y - z) - (3x + 2y) - (x - 2y + 3z)$. $-2x - 3y - 2z$
5. $2\frac{1}{2}x - 4\frac{1}{3}y - (5\frac{1}{4}y + 3z) - (1\frac{1}{6}x - 4\frac{3}{4}y - 4\frac{1}{5}z)$. $1\frac{1}{3}x - 4$ [illegible]

47. If there are brackets within brackets (Art. 40), *all the brackets may be successively removed by the preceding rule.* It is usual to *begin* with the innermost brackets.

Ex. 1 .

$$\begin{aligned} & 11x - [7x - \{8x - (9x - 6x)\}] \\ &= 11x - [7x - \{8x - 9x + 6x\}] \\ &= 11x - [7x - 8x + 9x - 6x] \\ &= 11x - 7x + 8x - 9x + 6x \\ &= 25x - 16x \\ &= 9x. \end{aligned}$$

Ex. 2 . $2a - [3b - \{-2c - (2a - \overline{2c + 3b})\}]$

$= 2a - [3b - \{-2c - (2a - 2c - 3b)\}]$

$= 2a - [3b - \{-2a + 3b\}]$, by striking out $2c - 2c$.

$= 2a - [+2a]$, by striking out $+3b - 3b$.

$= 2a - 2a = 0$.

48. A much easier way is to remove the brackets simultaneously, and write the result in one line. This may be done by the law that a bracket makes *one* quantity of what it incloses, and the sign before it affects the inclosed quantity as one quantity.

Thus Ex. 2 , in last Art., may be regarded as a *binomial*, the first term being $2a$ and the second *all* that [ ] incloses ; so { } incloses a binomial, viz. $2c$, and *all that* ( ) incloses ; similarly ( ) incloses two terms, and finally the $\overline{}$ (vinculum) incloses two terms, $2c + 3b$. Hence,

1°. The − before [changes *two* signs, that of $3b$ and that of {.

2°. The sign before { being now +, the *two* signs in { do not change.

3°. The − before (changes two signs, that of $2a$ and that of the $\overline{}$.

4°. The sign before the $\overline{}$ being now +, its quantity does not change signs....In using this shorter method the learner may at first write down the *signs alone*, and afterwards insert the proper *terms*.

Ex. 1 . $2x - [3y - \{-2z - (2x - \overline{2z + 3y})\}]$.

$-\qquad -\quad -\qquad +\quad +$ ....Writing signs first.

i. e., $2x - 3y - 2z - 2x + 2z + 3y = 0$.

Ex. 2 . $3a^2 - [6a^2 - \{8b^2 - (9c^2 - 2a^2)\}]$.

$-\qquad +\qquad -\qquad +$ ....Writing signs first.

i. e., $3a^2 - 6a^2 + 8b^2 - 9c^2 + 2a^2 = -a^2 + 8b^2 - 9c^2$.

Ex. 3 . $84 - 7[-11x - 4\{-17x + 3(8 - \overline{9 - 5x})\}]$.

i. e., $84 + 77x - 28 \cdot 17x + 84 \cdot 8 - 84 \cdot 9 + 84 \cdot 5x = 21x$.

Here, note that the coefficient 7 multiplies 4, the coefficient of {, and that 28 (=7 × 4) multiplies 3, the coefficient of (, making it 84.

EXERCISE XIX.

Remove brackets and collect terms in the following :—

1. $(a + x) - (b - x) - (2a - 3b) - (a - b - x)$. $-2a + 3x + 3$

2. $3a - \{(a - 3c) - (2b - c)\} + (-a - b - c)$. $a + b + c$

3. $\{2a^2 - (3ab - b^2)\} - \{a^2 - (4ab + b^2)\} + \{2b^2 - (a^2 - ab)\}$. $2ab + c$

4. $\{2x - (3y - z)\} - \{y + (2x - z)\} + \{3z - (x - 2y)\} - \{2x - (y - z)\}$. $-3x -$

5. $1 - \{1 - (1 - 4x)\} + \{2x - (3 - 5x)\} + (-7x + 7)$. $5 - 4x$

6. $\{2a - (3b + c - 2d)\} - \{(2a - 3b) + (c - 2d)\} + \{2a - (3b + c)\}$. $2a - 3b - 3c + 4d$

7. $6a - [4b - \{-4a - (6a - 4b)\}]$. $-4a$

8. $[x - 5y - \{x - (5z - \overline{2z - y} - 4y) + 2x - (x - \overline{2y + z})\}]$. $-x - 10z +$

9. $x - \{2x - y - [3x - 2y - (4x - 3y)]\}$. $-2x + 2y$

10. $my - [x + 3y + \{2my - 3(x - y) - 4ab\} + 5]$. $2x - 6y - my + [illegible]$

11. $a - 2(b - c) - [-\{4a - b - c - 2(a + b + c)\}]$. $3a - 5b -$

12. $-2\{-[-(x - y)]\} + \{-2[-(x - y)]\}$. 0

13. $\frac{1}{2}x - \frac{1}{2}\left(\frac{2}{3}y - \frac{1}{2}z\right) - \left[x - \left\{\frac{1}{2}x - \left(\frac{1}{2}y - \frac{1}{4}z\right)\right\} - \left(\frac{2}{3}y - \frac{1}{2}z\right)\right]$. $-\frac{1}{6}y$

14. $\frac{1}{4}\{a - 5(b - a)\} - \frac{3}{2}\left[\frac{1}{3}\left(b - \frac{a}{3}\right) - \frac{2}{9}\left\{a - \frac{3}{4}\left(b - \frac{4a}{5}\right)\right\}\right]$. $\frac{11}{3}a - 2b$

15. Find value of $a - (b - c) - \{b - (a - c)\} - [a - \{2b - (a - c)\}]$ when $c = \frac{1}{2}[2 - \{2 - (2 - \overline{2 - x})\}]$.

16. If $a = 2$, $b = 3$, $x = 6$, $y = 5$, find the value of $a + 2x - \{b + y - [a - x - \overline{b - 2y}]\}$. 9

49. *Conversely*, by the rules of Art. 46, the terms of a quantity may be enclosed within brackets with the sign + or − preceding the bracket.

1°. If the sign + precedes, there is no change of signs.

2°. If the sign − precedes, the sign of every term enclosed in brackets is changed.

It is usual to place before the bracket the sign of the term which is to stand first *within* the bracket.

Ex. 1 . $x - y + z - a = (x - y) + (z - a)$

$$\begin{aligned}
&= (x + z) - (y + a)\\
&= (x + z) + (-y - a)\\
&= x - (y - z + a)\\
&= x + (z - y - a)\\
&= x - (a - z + y)\\
&= -(-x + y) - (-z + a).
\end{aligned}$$

Ex. 2 . In $ax^4 + bx^2 - 6 + 2bx - 5x^2 + 2x^4 - 3x$, bracket the coefficients of like powers of x with + before all the brackets.

$$(a + 2)\,x^4 + (b - 5)\,x^2 + (2b - 3)\,x - 6.$$

Ex. 3 . Place in brackets $c - 3e + 2d + 6f - 7g + 2n - 6x + y$.

i. In pairs, with three pairs preceded by the sign $-$;

$$(c + 2d) - (3e - 6f) - (7g - 2n) - (6x - y).$$

ii. With *four* terms in each bracket, and the sign $-$ before the 2d bracket;

$$(c + 2d - 3e + 6f) - (7g - 2n + 6x - y).$$

iii. All but first two in brackets, preceded by the sign $-$.

$$c + 2d - (3e - 6f + 7g - 2n + 6x - y).$$

EXERCISE XX.

1. Reduce the following expressions to the form $x -$ (bracketed quantity :—

$-(a+b)$ i. $x - a - b$; $-a + x - 3b + 2y$. $x-(a+3b-2y)$

$-(2m-2n)$ ii. $x - 2m + 2n$; $-3b + x + 2c + 5d$. $x-($[illegible]$)$

$(2m+3a-2b)$ iii. $x - 2m - (3a - 2b)$; $x + a - (b - c) + (m - n)$.

$x-(b-a-c-m+n)$

iv. $12 - a - b + c + x$; $x - (a + b) - (p - q) - (m - n)$.

$x-(a+b-c-12)$ $x-$ [illegible]

2. Collect in brackets the coefficients of x, y, and z in the following :—

$(2a-4b-3c)x-(6a+3c)y+(4b-ac)z$

i. $2ax - 6ay + 4bz - 4bx - 3cx - 3cy - acz$.

ii. $ax - by + cz - bx + cy - az + cx - ay + bz$, with a first in each quantity.

iii. $12ax - 12ay - 4by - 12bz - 15cx - 6cy - 3cz$.

ii $(a-b+c)x-(a+b-c)y-(a-b-c)z$

iii [illegible]

3. In the following quantities bracket coefficients of like powers of x so that $+$ shall stand before each bracket :—

i $2 + (7 - 2c)x + (5a - 3)x^2 + (9a - 7)x^3$

i. $2 - 7x^3 + 5ax^2 - 2cx + 9ax^3 + 7x - 3x^2$.

ii. $2cx^5 - 3abx + 4cx - 3bx^4 - a^2x^5 + ax^4 - mx^3 + x^3$.

iii. $2 - 7x + ax - 2x^2 + (b + 1)\,x^2 - (b - c)\,x^3 + x^3 + x^4 - ax^4$.

ii $(2c - a^2)x^5 + (a - 3b)x^4 + (1 - m)x^3 + (4c - 3ab)x$

x iii $(1 - a)x^4 + (1 - b + c)x^3 + (b - 1)x^2 + (a - 7)x + 2$

4. In the following expressions collect the coefficients of the powers of x with the sign $-$ prefixed to all brackets :—

i $-(3c^2 - 5a)x - (abc - 7)x^3 - (ab - 7)x^5$

i. $7x^3 - 3c^2x - abx^5 + 5ax + 7x^5 - abcx^3$.

ii. $1 - ax + bx^2 - x^2 + (c - 1)\,x^3 - ax^3 + (b - a)\,x^4 + x^4 + x$.

iii. $3b^2x^4 - bx - cx - ax^4 - 5c^2x^2 + cx^3 - x^3 - x^2$.

ii $1 - (a - 1)x - (1 - b)x^2 - (a - c + 1)x^3 - (a - b - 1)x^4$

iii $-(a - 3b^2)x^4 - (1 - c)x^3 - (1 + 5c^2)x^2 - (b + c)x$

5. Simplify and collect coefficients :—

$(a - c + 1)x^3 - (a + 2b + 1)x^2 + (b + c)x + 3$

i. $ax^3 - 2bx^2 + 3 + bx - cx^3 - x^2 + x^3 - ax^2 + cx$.

ii. $5ax^3 - 7\,(bx - cx^2) - \{6bx^2 - (3ax^2 + 2ax) - 4cx^3\}$.

iii. $ax^2 - (b + c)\,x + ab - bx^2 - (c + a)\,x - bc + cx^2 - (a + b)x - ca$.

x ii $(5a + 4c)x^3 + (7c - 6b + 3a)x^2 + (2a - 7b)x$

iii $(a - b + c)x^2 - 2(a + b + c)x + ab - bc - ca$

6. If $a = 25$, $b = 9$, $c = 4$, $d = 1$, find the values of the following :—

i. $2(\sqrt{a} + 4\sqrt{d}) - (2\sqrt{b} + 3\sqrt{c})$; 6

$\sqrt{(4a)} + \sqrt{(9b)} - \{\sqrt{(16c)} + \sqrt{(25d)}\}$. 6

ii. $3\sqrt{a} + \sqrt{(16d)} - \{2\sqrt{(4b)} + 4\sqrt{(9c)}\}$; −17

$\sqrt{a^2} - 2\sqrt[3]{b^3} - (3\sqrt[4]{c^4} + 4\sqrt[5]{d})$. −9

iii. $\sqrt[3]{(5a)} - \{2\sqrt[3]{(3b)} + \sqrt[3]{(2c)} - 4\sqrt[3]{d}\}$: +1

$\sqrt{(bc)} - \{3\sqrt{(acd)} + 4\sqrt{(b^2d)} - \sqrt{c^2d^2)}\}$. −56

7. Express in its simplest form

$$(b + c - a)\,x + (c + a - b)\,y + (a + b - c)\,z + (c + a - b)\,x + (a + b - c)\,y + (b + c - a)\,z + (a + b - c)\,x + (b + c - a)\,y + (c + a - b)\,z.$$

$(a + b + c)(x + y + z)$

8. Collect coefficients, and arrange in ascending powers of x,

$$(2x - 2p)\,x^2 - 2x^2 + (2c - 3r)\,x - 3a + (2p + 2a)\,x^3 - (b - q)\,x^2 - x + a - (b + q)\,x^2 + (2a - p)\,x^3 + (1 - c)\,x + 5a - x^3 + 3bx^2 - (c - 2r)\,x - 6a.$$

$-3a - px - (2 - b)x^2 + (4a - p - 1)x^3 + 2x^4$

9. Simplify, and arrange in powers of x,

$$x^5 - 4yx^4 - \frac{1}{6}\left[12zx - 4\left\{3yx^4 - 9\left(\frac{cx}{2} - yx^5\right) - \frac{3}{2}zx^4\right\}\right].$$

$(6y + 1)x^5 - (z + 2y)x^4 - (2z + 3c)x$

CHAPTER IV.

MULTIPLICATION.

50. When two or more quantities are to be multiplied together, the product may be *indicated* by enclosing the quantities in brackets and writing them in a row without the sign of multiplication. In the case of a monomial factor the bracket is usually omitted.

Ex. 1. The product of $x-2a$, $x+a$, and $-2a$, is represented by $-2a(x-2a)(x+a)$. It may also be represented thus, $(-2a)(x-2a)(x+a)$.

Ex. 2. $-2x \times +3y = -2x(+3y)$; $3x(-5y) = 3x \times -5y$.

51. It is convenient to consider three cases of Multiplication :—

I. **Monomials by Monomials;**

II. **Polynomials by Monomials;**

III. **Polynomials by Polynomials.**

N. B.—*The rules for signs, coefficients, and exponents are the same for all cases.*

52. I. Monomials by Monomials.—Three things have to be considered, viz. coefficients, exponents, and signs.

Coefficients and Exponents.—Since

$$\begin{aligned} 3a^3 &= 3 \times aaa, \text{ and} \\ 5a^2 &= 5 \times aa; \\ \therefore\ 3a^3 \times 5a^2 &= 3 \times aaa \times 5 \times aa \\ &= 3 \times 5 \times aaa \cdot aa, \end{aligned}$$

since the factors, as in Arithmetic, may be taken in any order,

$$= 15a^{3+2} = 15a^5.$$

So also,

$$3a^2b^3 \times 5a^3b = 3 \times 5 \times a^{2+3} \cdot b^{3+1} = 15a^5b^4;$$

and, generally,—

$$pa^m \times qa^n = p\cdot aaa\ldots\text{to } m \text{ factors} \times q\cdot aaa\ldots\text{to } n \text{ factors}$$
$$= pq \times aaa\ldots\text{to } m+n \text{ factors} = pqa^{m+n}.$$

53. Rule of Signs.—If the multiplier is *positive*, we have :—

1°. $+4\times(+3) = (+4)+(+4)+(+4) = +(+12) = +12$
So, $+a\times(+b) = (+a)+(+a)+$, etc. to b terms $= +(+ab) = +ab$.

2°. $-4 \times +3 = \quad (-4) + (-4) + (-4)$
$= +(-4 \quad -4 \quad -4) = +(-12) = -12.$

So, $-a\times +b = \quad (-a)+(-a)+(-a)$ to b terms
$= +(-a \quad -a \quad -a$ " " $) = +(-ab) = -ab.$

3°. If the multiplier is *negative*, it can mean only that the product of which it is a factor *is to be subtracted*, *i. e.*, all its **signs are to be changed.** Thus,

$$+4 \times (-3) = -(4+4+4) = -12.$$

So, $+a \times (-b) = -(a + a + a\ldots$to b terms$) = -(ab) = -ab.$

4°. So also, $-3 \times (-4)$, means that -3 is to be multiplied by 4, and the product made *negative*, *i. e.*, *subtracted.* Thus,

$$-3 \times (-4) = -\{(-3)\times 4\} = -\{-12\} = +12.$$

So, $\quad -a \times (-b) = -\{-a \times b\} = -\{-ab\} = +ab.$

*From these examples we deduce the Rule of Signs:—Factors with **like** signs give a **positive** product; factors with **unlike** signs give a **negative** product.*

54. Hence to multiply two monomials :—*Multiply the coefficients; affix all the literal parts, each with an exponent which is the sum of its exponents in the separate factors; prefix the sign + if the signs of the factors are* **alike,** *the sign − if they are* **unlike.**

EXERCISE XXI.

(*a*)

Multiply :—

1. a by b; a^2b by ab^2; $3xyz$ by $4x^2yz^3$.

2. $3x^2$ by $5x^3y$; $2abm$ by $3a^2m^2$; $-x^2y^3$ by $3ab$.

3. -3 by 4; -3 by -4; $-3ab$ by 4; $3ab$ by -4; $-ab$ by $-ab$.

4. $5a^2x$ by $-3a^2x^3$; $7ab^3$ by $-a^3b$; $5x^3y^3$ by $-3x$; $3a^5b$ by $-6ab^5$.

5. $\frac{1}{2}ab$ by $\frac{1}{3}a^2b^3$; $3a^2b^3c^4$ by $-4b^2c^3a^4$; $\frac{3}{2}m^5y$ by $-\frac{2}{3}m^2y^5$; $3pq^2$ by $-apq$.

(*b*)

1. If $a=-2$, $b=3$, $c=-1$, find the values of :—

$3a^2b$; $8abc^2$; $-5c^3$; $-b^2c^2$; $-7a^3bc$; $-5a^2b^2c^2$.

2. Write out the following products :—

i. $mx \times my \times mz$; $ax \times bx \times cx$; $2ab \times 3ab \times (-4ab)$.
ii. $3a^2m \times (-4a^2m) \times 3a^2m^2$; $a^2x^2 \times (-b^2y^2) \times c^2z^2 \times abcz$.
iii. $2abx \times (-7abx)$; $6x^2yz^2 \times (-3xyz^2)$; $xyz \times (-5x^2y^2z)$.

3. If $a=-4$, $b=-3$, $c=-1$, $f=0$, $x=4$, $y=1$, find the values of :—

i. $3a^2+bx-4cy$; $2ab^2-3bc^2+2fx$; $2a^2-3b^2+7cy^4$; a^3-b^3.
ii. $3a^2y^3-5b^2x-2c^3$; $-7a^2y \times +3a^2y^2 \times ax$;
$f(a^2+b^2-c^2-x^2)$.
iii. $\sqrt{ac}-\frac{3}{2}\sqrt{xy}+\frac{1}{2}\sqrt{b^2c^4}$; $7\sqrt{a^2x}-3\sqrt{b^4c^2}+5\sqrt{(a^2+b^2)}f$.

55. II. Polynomials by Monomials.—We have, from Arithmetic,

$$(8+3)\times 6 = 8\times 6 + 3\times 6 = 48+18 = 66;$$
$$(8-3)\times 6 = 8\times 6 - 3\times 6 = 48-18 = 30.$$

So, with algebraic quantities,

$$(a+b)\,c = a\times c + b\times c = ac+bc; \text{ and}$$
$$(a-b)\,c = a\times c - b\times c = ac-bc.$$

Therefore, to multiply a polynomial by a monomial :—*Multiply every term of the polynomial by the monomial multiplier.*

EXERCISE XXII.

(*a*)

1. Write down the products in the following cases :—

$3x^2-4xy-5y^2$ by $-4x$; x^2-xy+y^2 by $3x$; a^2+b^2 by ab.

2. Multiply $2a^2-2b^2$ by $\frac{1}{2}ab$; $5a^2y^3-7a^3y^2-7a^4y$ by $8ab$.

3. Multiply $x+y-z$ by xyz; $-x-y-z$ by $-4z$; $1+x$ by $-x^2$.

4. $5a^2+3b^2-2c^2$ by $4a^2bc^3$; $6a^3-5a^2b+7ab^2$ by $8a^2b^3$.

(*b*)

Perform the multiplications in the following cases :—

1. $-a^2bc+b^3-c^3$ by $-ab$; $3x-2y-4$ by $-\frac{5}{6}x$. $-\frac{5}{2}x^2+\frac{5}{3}xy+\frac{5}{3}x$
$a^3b^2c-ab^4+abc^3$
2. $-3ax(-by-2cz-5)$; $(3x^2-4y^2+5z^2)\times 3x^2yz^2$. $9x^4yz^2-12x^2y^3z^2+15x^2$
$3abxy+6acxz+15ax$
3. $(3x^3+2x^2y-7xy^2+y^3)\times(-5x^2y)$; $(-3-2ab+a^2b^2)\times(-a^4)$.
$-15x^5y-10x^4y^2+35x^3y^3-5x^2y^4$ $3a^4+2a^5b-a^6b^2$
4. $(-x+y^2-z^3+4x^2y^2z^2)\times(-3x^2y^2z)$; $(\frac{2}{3}a-\frac{1}{6}b-c)\times\frac{3}{8}ax$.
$3x^3y^2z-3x^2y^4z+3x^2y^2z^4-12x^4y^4z^3$; $\frac{1}{4}a^2x-\frac{1}{16}abx-\frac{3}{8}acx$
5. $(\frac{6}{7}a^2x^2-\frac{3}{2}ax^3-\frac{3}{7}a^3x^3)\times-\frac{7}{3}a^3x$; $-\frac{4}{7}x^5y^5(\frac{7}{4}x^3-\frac{4}{7}y^3)$.
$-2a^5x^3+\frac{7}{2}a^4x^4+a^6x^4$; $-x^8y^5+\frac{16}{49}x^5y^8$
6. $(\frac{1}{2}x^2-xy+\frac{1}{2}y^2-\frac{2}{3}xyz)\times\frac{3}{2}x^2y^2z^2$; $-\frac{5}{3}a^2x^2(-\frac{3}{2}a^2+ax-\frac{3}{5}x^2)$.
$\frac{3}{4}x^4y^2z^2-\frac{3}{2}x^3y^3z^2+\frac{3}{4}x^2y^4z^2-x^3y^3z^3$; $\frac{5}{2}a^4x^2-\frac{5}{3}a^3x^3+a^2x^4$
7. $\{(a+b)^3-(a+b)\}\times-2(a+b)$; $\{\frac{1}{2}(a-b)+\frac{1}{3}(a-b)^3\}$
$\times-6(a-b)^2$. $-2(a+b)^4+2(a+b)^2$; $-3(a-b)^3-2($ [illegible]
8. $\{(m^2-n)^3+(m^2-n)\}\times(m^2-n)^4$; $\{3(a+b)+2(a+b)^4\}(a+b)^n$.
$(m^2-n)^7+(m^2-n)^5$; $3(a+b)^{n+1}+2(a+b)^{n+4}$
9. $\{(a+b)^n+(a+b)^m\}\times(a+b)$; $\{(a-b)-(a-b)^2\}\times(a-b)^n$.
$(a+b)^{n+1}+(a+b)^{m+1}$; $(a-b)^{n+1}-(a-b)^{n+2}$

56. III. Polynomials by Polynomials.—We may multiply $(7+4-3)$ by $(8+3)$ thus :—

$(7+4-3)\times 8 = 8\times 7+8\times 4-8\times 3 = 8$ times the multiplicand.
$(7+4-3)\times 3 = 3\times 7+3\times 4-3\times 3 = 3$ " " "

Adding these products we have the complete product.

Similarly, with algebraic quantities :—

$$\begin{aligned}(a+b-c)\times(m+n) &= (a+b-c)\,m\\ &\quad+(a+b-c)\,n\\ &=(ma+mb-mc)\\ &\quad+(na+nb-nc).\\ &=ma+mb-mc+na+nb-nc.\end{aligned}$$

So also: $(a+b-c)(m-n) = m(a+b-c)-n(a+b-c)$
$= (ma+mb-mc)-(na+nb-nc)$

i. e., n times the Multiplicand is to be *subtracted* from m times the same : $\therefore$ the result is

$$ma+mb-mc-na-nb+nc.$$

Hence, *Multiply every term in the multiplicand by every term in the multiplier, observing the rule of signs as in the previous cases.*

Ex. 1. Multiply $3x - 4y$ by $4x - 2y$.

$3x - 4y$
$4x - 2y$
$12x^2 - 16xy$ .... $= 4x$ times the multiplicand.
$\quad - 6xy + 8y^2$ .... $= 2y$ " " " *subtracted.*
$12x^2 - 22xy + 8y^2$ .... $=$ *product* $= (4x - 2y)$ times the multiplicand.

Ex. 2. Multiply $7x^3 - 6x^2 + 5x - 4$ by $3x^2 - 4x - 5$.

$7x^3 - 6x^2 + 5x - 4$
$3x^2 - 4x \;\; - 6$
$21x^5 - 18x^4 + 15x^3 - 12x^2$ ... $= 3x^2$ times the multplcnd.
$\quad - 28x^4 + 24x^3 - 20x^2 + 16x$ ... $= -4x$ " "
$\qquad - 42x^3 + 36x^2 - 30x + 24$ .. $= -6$ " "
$21x^5 - 46x^4 - \;\; 3x^3 + \;\; 4x^2 - 14x + 24$ .. $=$ required product.

Ex. 3. Multiply $x^2 - ax + bx - ab$ by $x - c$.

$$
\begin{array}{l}
x^2 - ax \;\; + bx \;\; - ab \\
x \;\; - c \\
\hline
x^3 - ax^2 + bx^2 - abx \\
\quad - cx^2 \qquad\quad + acx - bcx + abc \\
\hline
x^3 - (a - b + c)\,x^2 - (ab - ac \;\; + bc)\,x + abc.
\end{array}
$$

N. B.—If more than two factors are to be multiplied together, find, first, the product of two of them; then, of that product and a third; and so on.

EXERCISE XXIII

Multiply :—

1. $2x - 3y$ by $3x - 2y$.
$6x^2 - 13xy + 6y^2$
2. $3x^2 - b$ by $5x - y$.
$15x^3 - 3x^2y - 5bx + by$
3. $x^2 - 3ax$ by $x + 3a$.
$x^3 - 9a^2x$
4. $2b^2 + 3ab$ by $- 5b^2 + 7a$.
$-10b^4 - 15ab^3 + 14ab^2 + 21a^2b$
5. $a^2 - ab + b^2$ by $a + b$.
$a^3 + b^3$
6. $a^2 + ab + b^2$ by $a - b$.
$a^3 - b^3$
7. $a^4 + a^2b^2 + b^4$ by $a^2 - b^2$.
$a^6 - b^6$
8. $y^3 - 3y + 2$ by $y^2 - 2$.
$y^5 - 5y^3 + 2y^2 + 6y - 4$
9. $a^2 + 6ab + \frac{1}{3}b$ by $a - \frac{1}{3}b$.
$a^3 + \frac{17}{3}a^2b + \frac{1}{3}ab$ [illegible]
10. $ax^n - bx^n$ by $ax + bx$.
$a^2x^{n+1} - b^2x^{n+1}$
11. $x^4 + x^3a + x^2a^2 + xa^3 + a^4$ by $x - a$. $x^5 - a^5$
12. $1 + x + x^2 + x^3 + x^4 + x^5$ by $1 - x$. $1 - x^6$

13. $y^3 - 2y^2 + 3y - 4$ by $y^3 + 2y^2 + 3y + 4$. $y^6 + 2y^4 - 7y^2 - 16$

14. $\frac{1}{2}x^2 - xy + y^2$ by $\frac{1}{2}x^2 + xy + y^2$. $\frac{1}{4}x^4 + y^4$

15. $m + nx + px^2$ by $a - bx$. $am + (an - bm)x + (ap - bn)x^2 - b$

16. $a + bx + cx^2$ by $1 - ax + bx^2$. $a - (a^2 - b)x +$ [illegible] $- (ac - b^2)x^3 + bcx^4$

17. $x^5 - 3x^3 + 4x - 2$ by $x^3 - 3x^2 - 3$.
$x^8 - 3x^7 - 3x^6 + 6x^5 + 4x^4 - 5x^3 + 6x^2 - 12x - 6$

18. $a^2 + b^2 + c^2 - ab + bc + ca$ by $a + b - c$.
$a^3 + b^3 - c^3 + 3abc$

19. $x^2 - xy + x + y^2 + y + 1$ by $x + y - 1$. $x^3 + y^3 + 3xy - 1$

20. $6x^6 - x^5 + 2x^4 - 2x^3 + 2x^2 + 19x + 6$ by $3x^2 + 5x + 1$.
$18x^8 + 27x^7 + 7x^6 + 3x^5 - 2x^4 + 65x^3 + 115x^2 + 49x + 6$

21. $x + y + z + a$ by $x + y - z - a$.
$x^2 + 2xy + y^2 - z^2 - 2az - a^2$

22. $4a + 3b + 2c + d$ by $4a + 3b - 2c - d$.
$16a^2 + 24ab + 9b^2 - 4c^2 - 4cd - d^2$

23. $4a - 3b + 2c - d$ by $4a - 3b - 2c + d$.
$16a^2 - 24ab + 9b^2 - 4c^2 + 4cd - d^2$

24. $x^2 + x - y^2 + y$ by $x^2 + x + y^2 - y$.
$x^4 + 2x^3 + x^2 - y^4 + 2y^3 - y^2$

25. $a^2 + 4b^2 + 9c^2 - 2ab + 6bc + 3ac$ by $a + 2b - 3c$.
$a^3 + 8b^3 - 27c^3 + 18abc$

26. $27x^3 - 36ax^2 + 48a^2x - 64a^3$ by $3x + 4a$.
$81x^4 - 256a^4$

27. $-2x^3y + y^4 + 3x^2y^2 + x^4 - 2xy^3$ by $x^2 + y^2 + 2xy$.
$x^6 + 2x^3y^3 + y^6$

28. $\frac{3}{4}x^2 - \frac{1}{2}ax + \frac{1}{3}a^2$ by $\frac{3}{2}x^2 - ax - \frac{2}{3}a^2$. $\frac{9}{8}x^4 - \frac{3}{2}ax^3 + \frac{1}{2}a^2x^2 -$

29. $(x + a)(x - a) \times (x^2 + a^2)(x^4 + a^4)$. $x^8 - a^8$

30. $ax^m + bx^n - 1$ by $ax^2 + bx^3$.
$a^2x^{m+2} + abx^{n+2} + abx^{m+3} + b^2x^{n+3} - bx^3 - ax^2$

31. $x^m + x^my^m + y^{2m}$ by $x^m - y^m$.
$x^{2m} + x^{2m}y^m - x^my^m - y^{3m}$

32. $(x - ab) \times (x + ab) \times (x^2 + a^2b^2) \times (x^4 + a^4b^4)$. $x^8 - a^8b^8$

33. $(x - 1) \times (x + 1) \times (x^2 + 1) \times (x^4 + 1) \times (x^8 + 1)$. $x^{16} - 1$

34. $(x^2 + ax + a^2)(x^2 - ax + a^2)(x^4 - a^2x^2 + a^4)$. $x^8 + x^4a^4 +$

35. $(x + y + 1)(x - y - 1)(x^2 + y^2 + 2y + 1)$.
$x^4 - y^4 - 4y^3 - 6y^2 - 4y - 1$

57. Detached Coefficients.—The labour of multiplying may be lessened by using only the coefficients. The quantities must be arranged in *ascending* or *descending powers, and the coefficients used in corresponding order*, those of the multiplicand in a horizontal line, those of the multiplier in a vertical column to the left. If *any powers* are wanting, these must be supplied by **Zero** terms.

Ex. 1. Multiply $7x^3 - 6x^2 + 5x - 4$ by $3x^2 - 4x - 6$.

$$\begin{array}{r|rrrrrrl}
 & 7 & -6 & +5 & -4\ldots\ldots & & & \text{coefficients of multiplicand.} \\
3 & 21 & -18 & +15 & -12\ldots\ldots & & & \text{'' '' '' into } 3 \\
-4 & & -28 & +24 & -20 & +16\ldots\ldots & & \text{'' '' '' '' } -4 \\
-6 & & & -42 & +36 & -30 & +24\ldots\ldots & \text{'' '' '' '' } -6 \\
\hline
 & 21 & -46 & -3 & +4 & -14 & +24 & \text{'' '' product.}
\end{array}$$

Now since $x^3 \times x^2 = x^5$, the first term will contain x^5, the 2d, x^4, the 3d, x^3, etc.; inserting these, we have:

$$21x^5 - 46x^4 - 3x^3 + 4x^2 - 14x + 24.$$

Note.—Compare the above with the work of the same question, Ex. 2, in last article.

Ex. 2. Multiply $2x^4 - 2x^2 - 2x + 3$ by $x^3 - 2x + 3$.

Here the *third power* of x, or (x^3), is wanting in the multiplicand, and the *second power*, (x^2), in the multiplier; we therefore supply a *zero* term in each case, thus:—

$$\begin{array}{r|rrrrrrrrl}
1 & 2 & +0 & -2 & -2 & +3\ldots\ldots & & & & \text{product from } 1 \\
+0 & & +0 & +0 & -0 & -0 & +0\ldots\ldots & & & \text{'' '' } 0 \\
-2 & & & -4 & -0 & +4 & +4 & -6\ldots\ldots & & \text{'' '' } -2 \\
+3 & & & & 6 & +0 & -6 & -6 & +9\ldots\ldots & \text{'' '' } 3 \\
\hline
 & 2 & +0 & -6 & +4 & +7 & -2 & -12 & +9\ldots\ldots & \text{coefficients of product.}
\end{array}$$

Highest power to be supplied is x^7, next x^6, etc. Therefore product is $2x^7 + 0x^6 - 6x^5 + 4x^4 + 7x^3 - 2x^2 - 12x + 9$; of course, the $0x^6$ may be omitted. Observe that when the coefficient of the 1st term of the multiplier is unity, the *coefficients of the multiplicand may be used as the coefficients of the first product.*

Ex. 3. Multiply $a - bx - cx^4$ by $1 - 2x + 3x^2$. We have

$$\begin{array}{r|lllllll}
1 & a & -b & +0 & +0 & -c & & \\
-2 & & -2a & +2b & -0 & -0 & +2c & \\
+3 & & & +3a & -3b & +0 & +0 & -3c \\
\hline
\end{array}$$

$a-(2a+b)+(3a+2b)-3b-c+2c-3c\ldots$ coefficients of the product,—which is, therefore,

$$a - (2a + b)x + (3a + 2b)x^2 - 3bx^3 - cx^4 + 2cx^5 - 3cx^6.$$

EXERCISE XXIV.

(*a*)

Using detached coefficients find the products of the following :—

1. $x^3 - 3x^2 + 3x - 1$ and $x^2 - 2x + 1$. $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$
2. $x^3 + 3x^2 + 3x + 1$ and $x^3 - 3x^2 + 3x - 1$. $x^6 - 3x^4 + 3x^2 - 1$
3. $x^3 - 7x^2 + 5x + 1$ and $2x^2 - 4x + 1$. $2x^5 - 18x^4 + 39x^3 - 25x^2 + x + 1$
4. $x^3 - 2x^2 + 3x - 4$ and $4x^3 + 3x^2 + 2x + 1$. $4x^6 - 5x^5 + 8x^4 - 10x^3 -$ [illegible] $- 5x$
5. $7x^5 - 5x^2 + 2$ and $3x^3 + 2x^2 - 7x + 1$. $21x^8 + 14x^7 - 49x^6 - 8x^5 - 10x^4 + 41x^3 - x^2 - 14x + 2$
6. $3x^4 - 2x^3 - 2x + 3$ and $x^2 + 3x - 2$. $3x^6 + 7x^5 - 12x^4 + 2x^3 - 3x^2 + 13x - 6$
7. $2x^3 - 4x^2 + 2x - 3$ and $2x^3 - 3$. $4x^6 - 8x^5 + 4x^4 - 12x^3 + 12x^2 - 3x + 9$
8. $x^4 - 2x^2 + 1$ and $x^4 - x^2 + 3$. $x^8 - 3x^6 + 6x^4 - 7x^2 + 3$
9. $x^3 + 5x^2 - 16x - 1$ and $x^3 - 5x^2 - 16x + 1$. $x^6 - 57x^4 + 266x^2 - 1$
10. $6x^6 - x^5 + 2x^4 - 2x^3 + 2x^2 + 19x + 6$ and $3x^2 + 4x + 1$. $18x^8 + 21x^7 + 6x^6 + x^5 + 6$[illegible]$x^4 +$ [illegible] $x^2 + 43x + 6$
11. $x^3 - 3x^2a + 3xa^2 - a^3$ and $x^3 + 3x^2a + 3xa^2 + a^3$. $x^6 - 3x^4a^2 + 3x^2a^4 - a^6$
12. $1 + x + x^2 + x^3 + x^4 + x^5 + x^6$ and $1 - x$. $1 - x^7$
13. $2 - 3a - a^2 + 2a^3$ and $2a^3 - a^2 - 3a + 2$. $4 - 12a + 5a^2 + 15a^3 - 11a^4 - 4a^5 + 4a^6$
14. $1 + x + x^2 + x^3 + x^4$ and $1 - x^2 + x^3 - x^7 + x^8 - x^{12} + x^{13}$. $1 + x + x^3 + x^4 + x^{17}$
15. $kx^3 + lx^2 + mx + n$ and $ax^2 + bx + c$.
$akx^5 + (al + bk)x^4 + (am + bl + ck)x^3 + (an + bm + cl)x^2 + (bn + cm)x + cn$

(*b*)

1. Show that $(a+b)^3 - (b+c)^3 - (a-c)^3 = 3(a+b)(b+c)(a-c)$.

2. What does $a^3 + b^3 - c^3 + 3abc$ become when $c = a + b$? 0

3. What does $x^4 + 12x^3 + 47x^2 + 66x + 28$ become when $y - 3$ is substituted for x? $y^4 - 7y^2 + 10$

4. If $2s = a + b + c$ show that $(s-a)^2 + (s-b)^2 + (s-c)^2 + s^2 = a^2 + b^2 + c^2$.

5. Find the continued product of $3x+7$, $3x-7$, and $81x^4 + 441x^2 + 2401$. $729x^6 - 1764$ [illegible]

6. Find the result when $a - n - p$ is put for x in $x^2 - (2an - n)x + a(a-n)$. $2a^2 - 2ap - 2a^2n + p^2 + 2anp$ [illegible]

7. What does $a^3 - b^3 - c^3 - 3abc$ become when $a - b - c = 0$? 0

CHAPTER V.

DIVISION.

58. In Multiplication we have *two factors* given to find their *product;* in Division we have the *product* and *one of the factors* given to find the *other factor.* In Algebra, as in Arithmetic, the given *product* is called the **Dividend,** the given factor is called the **Divisor,** and the required factor is called the **Quotient.**

The division of one quantity by another is *represented* by writing the dividend as numerator and the divisor as denominator of a fraction, or by enclosing the quantities in brackets and writing the sign of division between them. If one of the quantities is a monomial, it need not be bracketed.

59. As in Multiplication, so in Division it is convenient to make three cases :—I. When both Divisor and Dividend are Monomials. II. When the Dividend is a Polynomial and the Divisor a Monomial. III. When both Dividend and Divisor are Polynomials.

60. I. Monomial Division.—We have, arithmetically,

$$30 = 6 \times 5,$$

and therefore $$\frac{30}{5} = \frac{5 \times 6}{5} = \frac{1 \times 6}{1} = 6;$$

the common factor 5 being cancelled in the divisor and dividend. So also,

$$\frac{32}{8} = \frac{2^5}{2^3} = \frac{2 \cdot 2 \cdot 2 \times 2 \cdot 2}{2 \cdot 2 \cdot 2} = 2 \cdot 2 = 2^2 = 2^{5-3}.$$

Again, $$\frac{8}{32} = \frac{2^3}{2^5} = \frac{2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 2 \cdot 2 \cdot 2} = \frac{1}{2 \cdot 2} = \frac{1}{2^2} = \frac{1}{2^{5-3}}.$$

1°. So, algebraically, since

$$a^2 \times a^3 = a^5 = a \cdot a \times a \cdot a \cdot a,$$

$$\therefore\ a^5 \div a^3 = \frac{a \cdot a \times a \cdot a \cdot a}{a \cdot a \cdot a} = a \cdot a = a^2 = a^{5-3}.$$

2°. And generally, when $m > n$,

$$a^m \div a^n = \frac{a \cdot a \cdot a \ldots \text{to } m \text{ factors}}{a \cdot a \cdot a \ldots \text{to } n \text{ factors}} = \frac{a \cdot a \cdot a \ldots \text{to } (m-n) \text{ factors}}{1}$$

$$= \frac{a^{m-n}}{1} = a^{m-n}.$$

If $m < n$, then

$$a^m \div a^n = \frac{a \cdot a \cdot a \ldots \text{to } m \text{ factors}}{a \cdot a \cdot a \ldots \text{to } n \text{ factors}} = \frac{1}{a \cdot a \cdot a \ldots \text{to } (n-m) \text{ factors}}$$

$$= \frac{1}{a^{n-m}}.$$

3°. Since

$$3a^2 \times 5a^3 = 3 \cdot 5 \cdot a^{2+3} = 15a^5,$$

$$\therefore\ 15a^5 \div 5a^3 = \frac{3 \cdot 5 \cdot a^5}{5 \cdot a^3} = 3a^{5-3}.$$

And similarly,

$$3a^2 \div 15a^5 = \frac{3 \cdot a \cdot a}{3 \cdot 5 \cdot a \cdot a \cdot a \cdot a \cdot a} = \frac{1}{5 \cdot a \cdot a \cdot a} = \frac{1}{5a^{5-2}}.$$

Thus, *generally*, if $m > n$,

$$ax^m \div bx^n = \frac{a \cdot x \cdot x \cdot x \text{ to } m \text{ factors}}{b \cdot x \cdot x \cdot x \text{ to } n \text{ factors}} = \frac{a}{b} \cdot x \cdot x \ldots \text{to } m-n \text{ factors},$$

Or, if $m < n$,

$$= \frac{a}{b} \cdot \frac{1}{x \cdot x \ldots \text{to } n-m \text{ factors}}.$$

In Division, therefore, we *divide* coefficients and *subtract* exponents. (See Art. 54.)

N. B.—$a^n \div a^n = 1$; but $a^n \div a^n = a^{n-n} = a^0$; $\therefore a^0 = 1$, which corresponds with Art. 30.

61. If *all* the factors of the divisor are factors of the dividend, the division is *exact*. If the divisor contains a factor, or factors, not found in the dividend, the division is *inexact*.

62. Hence, for Monomial Division: *Write the dividend and divisor in the form of a fraction, and remove the factors which are common to both.*

63. It is evident that the **Rule of Signs** in Division corresponds to that in Multiplication. That is, *If divisor and dividend have the* **same** *sign, the quotient is* **positive**; *if they have* **opposite** *signs, the quotient is* **negative**; for,

1°. $+x \times (+m) = +mx$; $\therefore +mx \div (+m) = +x$.

2°. $-x \times (+m) = -mx$; $\therefore -mx \div (+m) = -x$.

3°. $+x \times (-m) = -mx$; $\therefore -mx \div (-m) = +x$.

4°. $-x \times (-m) = +mx$; $\therefore +mx \div (-m) = -x$.

EXERCISE XXV.

Perform the following divisions :—

1. $3x^3 \div x^2$; $21x^5 \div 3x^4$; $-21x^5 \div 7x^3$; $-35x^6 \div 7x^4$.
2. $a^5c^6 \div -a^2c^2$; $a^6x^3 \div a^2x^3$; $28a^4b^4c^4 \div -4a^2b^2c^2$.
3. $33a^2mx \div 11mx$; $-24x^2yz \div 12xyz$; $4x^m \div -2x^3$.
4. $-27a^7b^7c \div -9a^2b^2$; $-x^2y^2 \div -\frac{4}{7}xy$; $-\frac{1}{10}x^4y \div \frac{1}{5}x^3y$.
5. $-13a^3bc^2 \div 13abc$; $a^{p+1} \div -a^2$: $ax^{n+1} \div x^{n-3}$.
6. $6a^mb^n \div -3ab$; $-a^n \div a^{n-1}$; $ma^{3+n} \div a^{n+2}$; $6x^{p+2} \div -3x^{-p}$.
7. $-18a^mp^n \div -6a^np$: $44a^5(x-y)^n \div 11a^3(x-y)^3$;
   $(a+b)^n \div -(a+b)^m$.
8. $4a^2m^4x^5 \div -5a^5m^3x$; $-3a^2b^3c^4 \div 4ab^2c^5$;
   $3a^2b^2c^4 \times 4a^2b^2 \div a^3b^4c^5$.
9. $-ab \cdot bc \cdot ca \div b^3$; $a^mb^n \div -ab^n$; $x^ny^4 \div x^4y^n$, $(n > 4)$.
10. $mx^{m+1}y \div nxy^{n+1}$; $ca \cdot ab \cdot bc \cdot abc \div (ax \cdot bx \cdot cx)$.

64. II. Polynomials by Monomials.—When the divisor is a monomial and the dividend a polynomial: *Divide every term of the dividend by the monomial divisor; the required quotient will be the sum of the partial quotients.*

For since $(a+b-c) \times m = ma + mb - mc$,

$$\therefore (ma + mb - mc) \div m = a + b - c.$$

The *signs* will be determined as in the last case.

EXERCISE XXVI.

Divide :—

1. $x^2 - 2xy$ by x ; $x^3y - y^3x$ by $-xy$; $a^3b^3 - a^2b^2 \div ab^2$.
2. $2a^2 - 6a^3x - 8a^5x^2$ by $2a^2$; $12abx - 24a^2b^2x^3 \div -12abx$.
3. $a^2 - ab - ac$ by $-a$; $a^3 - a^2b - a^2b^2$ by $-a^2$; $\frac{1}{2}x^2y^2 - \frac{1}{3}xy$ by $2xy$.
4. $21am^2x^m - 14a^2m^4x^n + 2a^5m^2x^p$ by $-7amx^n$.
5. $(a+b)^3$ by $(a+b)^2$; $8(a-b)^6 \div 2(a-b)^4$; $a^n + a^{n-1}$ by a^3.
6. $4x^4y^4 - 8x^3y^2 + 6xy^2$ by $-3xy$; $-\frac{5}{2}x^2 + \frac{6}{5}xy - \frac{10}{3}x$ by $-\frac{5}{6}x$.
7. $a^{2n} + xa^n$ by $-a^n$; $a^{3n}x^n - a^{2n}x^{2n} + a^nx^{3n}$ by a^nx^n.
8. Simplify $\dfrac{12a^3x^2y^3}{2axy^2} - \dfrac{10a^3x^2y^3}{2a^2x^2y} + \dfrac{15a^2x^3y^2}{5x^2y} - \dfrac{16a^2x^3y^2}{4ax^3}$.
9. $8x^4y - 4x^3y^2 + 12x^2y^4 - 4x^2y^2$ by -4^2y^2.
10. $2a^2b^2 - 3b^2c^2 + 4b^2c^2$ by $5a^2b^2c^2$; $16x^m - 4x^{m-4} + 6x^{m-8}$ by $4x^{m-6}$.
11. $x^2z^2 - xyz^2 + 3xz^2$ by $-4xyz^2$; $a^4x - a^3x^2 + a^2x^3 - ax^4$ by x^4a.
12. $-3ax^4 + 5a^2x^2 - 6a^3x^2 - a^4$ by $-2a^3x^2$; $(a-b)^{m-1}$ by $(a-b)^{m-n}$.
13. $27(a+b)^5 - 18(a+b)^3 + 9(a+b)^2$ by $18(a+b)^2$.
14. $a^m(a-b)^n - a^n(a-b)^m$ by $a^n(a-b)^n$.
15. $10(x+y)^m(x-y)^n - 5(x+y)^p(x-y)^q$ by $5(x+y)^2(x-y)^2$.
16. $(a+b)^m(a-b)^n - (a+b)^n(a-b)^m$ by $(a+b)^3(a-b)^3$.

65. III. Polynomials by Polynomials.—This is similar to "Long Division" in Arithmetic, where we, in effect, separate the dividend into parts, obtain the quotients of the several parts, and add these partial quotients for the complete quotient.

Ex. 1. Divide 805 by 23.

$$\begin{array}{r}
23\ )\ 805\ (\ 35 \\
69 \\
\hline
115 \\
115 \\
\hline
\end{array}$$

In this operation what we have really done is this :—

$$\frac{805}{23}=\frac{690+115}{23}=\frac{690}{23}+\frac{115}{23}=30+5=35.$$

Similarly, in Algebra ; since

1°. $(x+4)\times(x+5)=(x+4)\times x+(x+4)\times 5=x^2+9x+20$;

$$\therefore\ \frac{x^2+9x+20}{x+4}=\frac{(x+4)(x+5)}{x+4}=\frac{(x+4)\times x}{x+4}+\frac{(x+4)\times 5}{x+4}=x+5.$$

2°. $(2x-3)(3x-2)=(2x-3)\cdot 3x-(2x-3)\cdot 2=6x^2-13x+6$;

$$\therefore\ \frac{x^2-13x+6}{2x-3}=\frac{(2x-3)(3x-2)}{2x-3}=\frac{(2x-3)\cdot 3x}{2x-3}-\frac{(2x-3)\cdot 2}{2x-3}=3x-2.$$

3°. And, *generally*,

$$(a+b)(c+d)=(a+b)\cdot c+(a+b)\cdot d=ac+bc+ad+bd\,;$$

$$\therefore\ \frac{ac+bc+ad+bd}{a+b}=\frac{(a+b)\cdot c}{a+b}+\frac{(a+b)\cdot d}{a+b}=c+d.$$

But since we cannot always so readily find the parts of the dividend which will contain the divisor exactly, we work thus :—

Ex. 1.

$$\begin{array}{r|l} x^2+9x+20 & \underline{x+4}\ldots\text{Divisor.} \\ \underline{x^2+4x\phantom{{}+20}} & x+5\ldots\text{Quotient.} \\ 5x+20 & \\ \underline{5x+20} & \end{array}$$

1°. Here we divide the first term, x^2, of the dividend by the first term (x) of the divisor, getting the partial quotient, x.

2°. We multiply the divisor by this partial quotient, getting x^2+4x.

3°. We subtract this product from the dividend, getting the *part* (remainder) $5x+20$.

4°. We divide the first term, $5x$, of this part by (x), the first term of the divisor, getting the partial quotient, 5.

5°. We multiply the divisor by this partial quotient, getting $5x+20$, and subtract this from the part (remainder) $5x+20$; there is no *part*, or remainder, left.

Now in this operation we have, in effect, separated the dividend into the parts $x^2 + 4x$, *i. e.* $(x + 4)\cdot x$ and $5x + 20$, *i. e.* $(x + 4)\cdot 5$: *these parts correspond exactly with the parts in Ex. 1. which gave the partial quotients x and 5, i. e.* $x + 5$ for complete quotient.

However long the work may be in any case, the principle and the method are exactly similar.

The divisor and the quotient are written under each other for convenience in multiplying.

Ex. 2.

$$\begin{array}{rrrrr|ll}
\multicolumn{5}{c}{\text{Dividend.}} & & \\
3x^4 & -4x^3 & +2x^2 & +3x & -1 & x^2-x+1 & .\ \text{Divisor.} \\ \cline{6-6}
3x^4 & -3x^3 & +3x^2 & & & 3x^2-x-2 & .\ \text{Quotient.} \\ \cline{1-3}
 & -x^3 & -x^2 & +3x & & \ldots\ldots & .\ \text{1st Remainder.} \\
 & -x^3 & +x^2 & -x & & & \\ \cline{2-5}
 & & -2x^2 & +4x & -1 & \ldots\ldots & .\ \text{2d Remainder.} \\
 & & -2x^2 & +2x & -2 & & \\ \cline{3-5}
 & & & 2x & +1 & \ldots\ldots & .\ \text{3d Remainder.}
\end{array}$$

Here the first term, $2x$, of the 3d remainder does not contain x^2 exactly, and the division cannot be carried on without fractions; we write the result just as we do in Arithmetic when there is a remainder, thus:

$$3x^2 - x - 2 + \frac{2x+1}{x^2-x+1}.$$

Ex. 3. Divide $4ab^3 + 51a^2b^2 + 10a^4 - 48a^3b - 15b^4$ by $4ab - 5a^2 + 3b^2$.

Here we first arrange both *dividend* and *divisor* in descending powers of a.

$$\begin{array}{rrrrr|l}
10a^4 & -48a^3b & +51a^2b^2 & +4ab^3 & -15b^4 & -5a^2+4ab+3b^2 \\ \cline{6-6}
10a^4 & -8a^3b & -6a^2b^2 & & & -2a^2+8ab-5b^2 \\ \cline{1-3}
 & -40a^3b & +57a^2b^2 & +4ab^3 & & \\
 & -40a^3b & +32a^2b^2 & +24ab^3 & & \\ \cline{2-5}
 & & 25a^2b^2 & -20ab^3 & -15b^4 & \\
 & & 25a^2b^2 & -20ab^3 & -15b^4 & \\ \cline{3-5}
\end{array}$$

66. Hence for Polynomial Division :—

Arrange dividend and divisor in ascending or descending powers of the same letter.

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient.

Multiply the divisor by this first term and subtract the product from the dividend.

Consider the remainder as a new dividend and proceed as before.

The RULE OF SIGNS *is the same as in the former cases.*

EXERCISE XXVII.

Divide :—

1. $x^2 + 12x + 35$ by $x + 5$. $x + 7$
2. $a^2 - 11a + 30$ by $a - 5$. $a - 6$
3. $6x^2 - 13x + 6$ by $2x - 3$. $3x - 2$
4. $a^2 - 49a + 600$ by $a - 25$.
5. $6x^2 - 7x - 3$ by $2x - 3$. $3x + 1$
6. $3x^2 - x - 14$ by $x + 2$.
7. $12x^2 - x - 6$ by $4x - 3$.
8. $12x^2 + x - 6$ by $3x - 2$. $4x + 3$
9. $6x^2 - 13xy + 6y^2$ by $2x - 3y$.
10. $7x^2 - 50x + 7$ by $7x - 1$. $x - 7$
11. $15x^2 + 14x - 8$ by $5x - 2$. $3x + 4$
12. $10x^2 - 17x + 3$ by $2x - 3$. $5x - 1$
13. $x^4 - y^4$ by $x^2 + y^2$.
14. $81x^4 - 16y^4$ by $9x^2 - 4y^2$. $9x^2 + 4y^2$
15. $-4xy - 15y^2 + 96x^2$ by $12x - 5y$.
16. $7x^3 + 96x^2 - 28x$ by $7x - 2$.
17. $100x^3 - 3x - 13x^2$ by $3 + 25x$.
18. $x^3 + 2x^2 - 2x - 1$ by $x - 1$.
19. $a^3 - 2a + 1$ by $a - 1$.
20. $a^3x^3 - 1$ by $ax - 1$.
21. $2ab + 6abc - 8abcd$ by $1 + 3c - 4cd$.
22. $2a^5 - 3ba^4 - 6ba^3 + 13b^2a^2 - 6b^3a$ by $2a - 3b$.
23. $x^4 + 4y^4$ by $x^2 + 2y^2 - 2xy$. $x^2 + 2xy + 2y^2$
24. $x^4 - 13x^2 + 36$ by $x^2 + 5x + 6$. $x^2 - 5x + 6$
25. $x^4 - 5x^3 + 11x^2 - 112x + 6$ by $x^2 - 3x + 3$. $x^2 - 2x + 2$
26. $48x^3 - 76ax^2 - 64a^2x + 105a^3$ by $2x - 3a$.
27. $6ax^6 + 6ax^2y^6 + 42a^2x^2$ by $ax + 5ax$.
28. $-15a^4 + 37a^2bd - 29a^2cf - 20b^2d^2 + 44bcdf - 8c^2f^2$ by $3a^2 - 5bd + cf$.
29. $3a^4 - 8a^2b^2 + 3a^2c^2 + 5b^4 - 3b^2c^2$ by $a^2 - b^2$.
30. $3x^4 + 14x^3 + 9x + 2$ by $x^2 + 5x + 1$. $3x^2 - x + 2$

31. $ab + 2a^2 - 3b^2 - 4bc - ac - c^2$ by $2a + 3b + c$. $a - b - c$

32. $15a^4 + 10a^3x + 4a^2x^2 + 6ax^3 - 3x^4$ by $3a^2 - x^2 + 2ax$. $5a^2 + 3x^2$

33. $qp^3 + 3p^2q^2 - 2pq^3 - 2q^4$ by $p - q$. $p^2q + 4pq^2 + 2q^3$

34. $x^4 + m^2x - 2mnx + m^2n - n^2$ by $x^2 + mx + n$.
$x^2 - mx + m^2 - n$; $(m^2 - m^3)x$

35. $x^{m+1} + x^my + xy^m$ by $x^m - y^m$. $x + y$; $y^{m+1} + 2xy^m$

36. $x^{3n} - 3x^{2n}y^n + 3x^ny^{2n} - y^{3n}$ by $x^n - y^n$. $x^{2n} - 2x^ny^n + y^{2n}$

37. $a^2x^3 + a^5 - 2abx^3 + b^2x^3 + a^3b^2 - 2a^4b$ by $ax - bx + a^2 - ab$.
$ax^2 - bx^2 - a^2x + abx + a^3 - a^2b$

67. Sometimes the work can be shortened by using **brackets**. Also, by using **detached coefficients** as described under multiplication. (Art. 57.)

Ex. 1 .
$$\begin{array}{l|l}
a^3 - 3abc + (b^2 - c^2) & a + (b + c) \\ \cline{2-2}
\underline{a^3 + (b + c)a^2} & a^2 - (b + c)a + (b^2 - bc + c^2) \\
\quad - (b + c)a^2 - 3bca & \\
\quad \underline{- (b + c)a^2 - (b + c)^2a} & \\
\qquad (b^2 - bc + c^2)a + (b^3 + c^3) & \\
\qquad \underline{(b^2 - bc + c^2)a + (b^3 + c^3)} &
\end{array}$$

Ex. 2 . Divide $x^7 + x^5 + x^4 + x^3 + x^2 + 1$ by $x^5 + x^4 + x^3 + x^2 + x + 1$.

Write the *coefficients*, inserting *zero* for missing powers, thus:

$$\begin{array}{l|l}
1 + 0 + 1 + 1 + 1 + 1 + 0 + 1 & 1 + 1 + 1 + 1 + 1 + 1 \text{ coeffic's of div'r.} \\ \cline{2-2}
\underline{1 + 1 + 1 + 1 + 1 + 1} & 1 - 1 + 1 \quad . \quad . \quad . \quad \text{`` \quad `` quo'ent.} \\
\;\; - 1 + 0 + 0 + 0 + 0 + 0 & \\
\;\; \underline{- 1 - 1 - 1 - 1 - 1 - 1} & \\
\qquad 1 + 1 + 1 + 1 + 1 + 1 & \\
\qquad \underline{1 + 1 + 1 + 1 + 1 + 1} &
\end{array}$$

$\therefore$ the quotient is $x^2 - x + 1$.

EXERCISE XXVIII.

Divide :—

✓ 1. $a^3 + b^3 - c^3 + 3abc$ by $a + b - c$. $a^2 + b^2 + c^2 - ab + bc + ca$

2. $x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc$ by $x - a$.
$x^2 - (b + c)x + bc$

✓ 3. $y^5 - my^4 + ny^3 - ny^2 + my - 1$ by $y - 1$.
$y^4 - (m - 1)y^3 - (m - n - 1)y^2$ [illegible] $+ 1$

✓ 4. $p^3 - q^3 + r^3 + 3pqr$ by $p - q + r$.

$p^2 + q^2 + r^2 + pq - qr$ [illegible]

$1 - x + 2y + x^2 + 2xy + 4y^2$

5. $1 + x^3 - 8y^3 + 6xy$ by $1 + x - 2y$; and $1 - x^3 + 8y^3 + 6xy$ by $1 - x + 2y$.

$1 + x - 2y + x^2 + 2xy + 4y^2$

6. $x^4 + y^4 - z^4 + 2x^2y^2 - 2z^2 - 1$ by $x^2 + y^2 - z^2 - 1$.

$x^2 + y^2 + z^2 + 1$

7. $3x^4 - 4x^3 + 2x^2 + 3x - 1$ by $x^2 - x + 1$.

$3x^2 - x - 2$, rem $2x + 1$

8. $8x^6 - 8x^5 + 12x^4 - 7x^3 + 3x^2 - 1$ by $4x^4 - 2x^3 + 3x^2 - x - 1$.

$2x^2 - x + 1$

68. Another method may often be used, which is the inverse operation of multiplication with detached coefficients (Art. 57), and is called **Horner's Synthetic Division.**

Ex. 1. Multiply $x^2 - 2x - 1$ by $x^2 + 3x - 3$.

$$
\begin{array}{r|lllll}
 & x^2 - 2x - 1 & & & & \\
\hline
x^2 & x^4 - 2x^3 - x^2 & \ldots\ldots & p_1 & \ldots\, \textit{First} \text{ partial product.} \\
+3x & \quad + 3x^3 - 6x^2 - 3x & \ldots & p_2 & \ldots\, \textit{Second} \text{ partial product.} \\
-3 & \qquad\quad - 3x^2 + 6x + 3 & \ldots & p_3 & \ldots\, \textit{Third} \text{ partial product.} \\
\hline
 & x^4 + x^3 - 10x^2 + 3x + 3 & \ldots & P & \ldots\, \textit{Complete} \text{ product.}
\end{array}
$$

Now, P is the sum of the three horizontal rows (p_1, p_2, p_3) between the lines; ∴ by *subtracting the second and third rows* (p_2, p_3) *from* P *we get the first row* (p_1); *and this row divided by the first term* (x^2) *of the multiplier will give the multiplicand.*

Hence to divide P by $x^2 + 3x - 3$, proceed thus :—

$$
\begin{array}{r|rrrrr}
x^2 & x^4 & +\ x^3 & -\ 10x^2 & +\ 3x & +\ 3 \\
\hline
-3x & & -\ 3x^3 & +\ 6x^2 & +\ 3x & \\
 & & & & & \\
+3 & & & +\ 3x^2 & -\ 6x & -\ 3 \\
\hline
 & x^2 & -\ 2x & -\ 1 & & \ldots\ldots \text{ Quotient.}
\end{array}
$$

1°. To the left of the dividend write the divisor with all its signs changed except that of the first term. *This is done to subtract the two rows corresponding to* p_3 and p_2 above.

2°. Divide the first term of the dividend by the first term of the divisor (x^2), getting x^2, the *first* term of the quotient, and multiply x^2 into $-3x$ and $+3$, placing the products in the *oblique column*, $-3x^3 + 3x^2$.

3°. Add the *second vertical* column, $-3x^3 + x^3$, divide by x^2 as before, getting $-2x$, the *second* term of the quotient, and multiply $-2x$ into $-3x + 3$, getting the oblique column, $+6x^2 - 6x$.

4°. Now add the third vertical column, ($+3x^2$, $+6x^2$, and $-10x^2$), getting $-x^2$, which divided by x^2 gives -1, the *third* term of the quotient; this, multiplied into $-3x+3$, gives the oblique column, $+3x-3$. The remaining two vertical columns added give each zero, therefore the division is exact.

By using detached coefficients the labor is still further lessened.

Ex. 2. Divide x^4+4x^2+16 by x^2-2x+4.

$$\begin{array}{r|rrrrr}
1 & 1 & +0 & +4 & +0 & +16 \\
+2 & & +2 & +4 & +8 & \\
-4 & & & -4 & -8 & -16 \\
\hline
 & 1 & +2 & +4 & +0 & +0
\end{array}$$

i. e., x^2+2x+4 . . . . Quotient.

Ex. 3. Divide $6x^5+5x^4-17x^3-6x^2+10x-2$ by $2x^2+3x-1$.

$$\begin{array}{r|rrrrrr}
2 & 6 & +5 & -17 & -6 & +10 & -2 \\
-3 & & -9 & +6 & +12 & -6 & \\
+1 & & & +3 & -2 & -4 & +2 \\
\hline
 & 3 & -2 & -4 & +2 & +0 & +0
\end{array}$$

i. e., $3x^3-2x^2-4x+2$ . . . . Quotient.

Ex. 4. Divide $5x^4-4x^3+3x^2-2x+1$ by x^2-3x+5.

$$\begin{array}{r|rrrrr}
1 & 5 & -4 & +3 & -2 & +1 \\
+3 & & +15 & +33 & +33 & \\
-5 & & & -25 & -55 & -55 \\
\hline
 & 5 & +11 & +11; & -24 & -54
\end{array} \quad \text{. . . . Remainder.}$$

i. e., $5x^2+11x+11$ . . . . Quotient.

$x^4 \div x^2 = x^2$. There will be, therefore, *three* terms in the quotient, and the remainder is $-24x-54$.

NOTE.—*If a vertical line be drawn with as many vertical columns to the* **right** *of it as are* **less** *by unity than the number of terms in the divisor, it will mark where the remainder begins to be formed.* Thus, in the last example, the vertical line would cut off $-24-54$ from the quotient line in the solution, and therefore the *remainder* is $-24x-54$.

Horner's Method is of special use when the divisor is a *binomial*.

EXERCISE XXIX.

Divide :—

1. $2x^5 + 7x^4 + 20x^3 + 30x^2 + 34x + 35$ by $x^2 + 2x + 5$.

$2x^3 + 3x^2$ [illegible]

2. $2x^4 - 7x^3 + 16x^2 - 16x + 8$ by $2x^2 - 3x + 2$. $x^2 - 2x$ [illegible]

3. $5x^5 - 18x^3 - 8x^2 + 20x - 5$ by $x^3 + 2x^2 - 3$.

$5x^2 - 10x + 2$; $3x^2 -$ [illegible] $+ 1$

4. $x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$ by $x^2 - 2x + 1$.

x [illegible] $3x^2 + 3x - 1$

5. $4x^5 - 7x^4 + 25x^3 - 15x^2 + 8x + 10$ by $x^2 - x + 5$.

$4x^3 - 3x^2 + 2x + 2$

6. $5x^4 - 7x^3 + 15x^2 - 12x - 36$ by $x^2 + x + 3$.

$5x^2 - 12x + 12$; $12x - 72$

7. $5x^5 - 15x^3 + 20x + 42$ by $x^3 - 2x^2 + 3$.

$5x^2 +$ [illegible] $+ 15$; $-5x^2 - 6x + 27$

8. $10x^{10} + 10x^6 + 10x^3 - 100$ by $x^7 + x^3 - x + 1$. $10x^3$; $10x$ [illegible]

9. $1 + 6x^5 + 5x^6$ by $1 + 2x + x^2$; and $a^6 - 6a + 5$ by $a^2 - 2a + 1$.

$1 - 2x + 3x^2 - 4x^3 + 5x^4$; $a^4 + 2a^3 + 3a^2$ [illegible]

10. $x^4 - 4xy^3 + 3y^4$ by $x^2 - 2xy + y^2$; and $m^4 + 4m + 3$ by $m^2 + 2m + 1$.

$x^2 + 2xy + 3y^2$; $m^2 - 2m + 3$

11. $x^8 - 2x^3 + 1$ by $x^2 - 2x + 1$; and $a^6 + 2a^3b^3 + b^6$ by $a^2 + 2ab + b^2$.

$x^4 +$ [illegible]

12. $3x^6 + 7x^5 - 12x^4 + 2x^3 - 3x^2 + 13x - 6$ by $x^2 + 3x - 2$.

$3x^4$ [illegible] $- 2x + 3$

13. $x^7 - 3x^6 + 4x^4 + 18x^3 - 7x + 12$ by $x^3 - 3x^2 + 3x - 1$.

$x^4 - 3x^2 - 6x + 15$; [illegible] $x^2 - 56x + 27$

14. $x^8 - 3x^7 - 5x^5 + 2x^4 + 5x^3 + 4x^2 + 1$ by $x^3 + 2x - 1$.

$x^5 - 3x^4 - 2x^3 + 2x^2 + 3x - 1$; [illegible]

15. $10x^6 - 11x^5 - 3x^4 + 20x^3 + 10x^2 + 2$ by $5x^3 - 3x^2 + 2x - 2$.

$2x^3 - x^2 - 2x + 6$; $2x$ [illegible] $+ 10$

16. $x^5 + 5x^4 + 10x^3 + 15x^2 + 5x + 1$ by $x + 1$.

$x^4 + 4x^3 +$ [illegible] $+ 6x - 6$

17. $4x^4 - 20x^3 + 32x^2 - 22x + 3$ by $2x^2 - 6x + 1$. [illegible]

18. $x^{12} + 2x^6 + 1$ by $x^4 + 2x^2 + 1$; and $x^5 + 1$ by $x + 1$.

$x^8 - 2x^6 +$ [illegible] $+ 1$; [illegible] $- x + 1$

19. $x^3 + (2a + b)x^2 + (a^2 + 2ab)x + a^2b$ by $x + a$; and $x^3 - 2ax^2 + (a^2 + b)x - ab$ by $x - a$.

$x^2 + ($[illegible]

20. $5y^4 + \frac{7}{2}ay^3 - \frac{107}{12}a^2y^2 + \frac{5}{6}a^3y + \frac{7}{6}a^4$ by $\frac{5}{2}y^2 + 3ay - \frac{7}{3}a^2$.

$2y^2 - ay -$ [illegible]

21. $5x^5 - 4x^4 + 3x^3 - 4x^2 + x + 4$ by $x + 4$. 3

$5x^4 - 24x^3$ [illegible]

22. $x^6 - 102x^5 + 100x^4 + 102x^3 - 99x^2 - 201x$ by $x - 101$. 2

$x^5 - x^4 - x$ [illegible] $+ 1$; [illegible]

23. $2x^5 + 401x^4 - 199x^3 + 399x^2 - 602x + 211$ by $x + 201$.

$2x^4 - x^3 +$ [illegible]

24. $x^9 - 3x^8 - 31x^7 + 25x^6 + 3x^5 - 15x^4 - 8x^3 + 19x^2 + 3x + 10$ by $3x^4 - 21x^3 + 9x - 6$.

$\frac{1}{3}x^5$ [illegible]

69. Positive and Negative Quantities.—The signs + and − are used in Arithmetic purely as *signs of operation.* But in Algebra it is found convenient to use them also with a more extended meaning, to express a certain contrariety, or opposition, in the character of the units themselves. In this sense they are called *signs of affection*, and the terms positive and negative then mean that any number of units of the one kind taken with the same number of units of the other kind neutralize each other, or give a result represented by 0; thus, $+a$ taken with $-a = 0$, where the words "taken with" are used instead of the sign + used in the arithmetical sense of addition, and the signs expressed denote contrariety, or opposition, in the character of the a units in each case.

This may be illustrated in the following examples:—

If a denote a certain number of yards, then $+a$ would express that number of yards measured in some particular direction, say *east*, and $-a$ would represent the same number of yards measured in the *directly contrary*, or *opposite*, direction, *i. e.*, *west;* so that $(+a) + (-a) = 0$, and represents the point from which measurement begins.

If f represent the mechanical effect of a certain force, then $+f$ will mean that this force acts in *one direction*, say to the right, and $-f$ will denote the same, or an equal force acting in the directly *opposite direction*, so that the two forces would produce rest; *i. e.*, $(+a) + (-a) = 0$.

In book-keeping, if $+d$ expresses \$$d$ placed in the *credit* column, then $-d$ means \$$d$ placed in the *debit* column, and, as before, the two entries neutralize each other. Also, if a man's *assets* are \$$d$, and his *debts*, \$$d$, we may express his financial standing by using the signs + and − in the algebraic sense, thus $(+d) + (-d) = 0$; and if his debts are $+c+d$, then $+d-c-d = -c$ represents the state of his affairs, showing that he is \$$c$ "*worse than nothing.*"

If $+d^\circ$ denotes the height of the mercury *above* the zero point in a thermometer, then $-d^\circ$ means a fall of the same number of degrees *below* the zero point. If the mercury rises d° and afterwards falls d°, the result would be represented by $(+d^\circ) + (-d^\circ) = 0$.

If the line AB be fixed at the point A, while the point B revolves in one direction through the point C, we may call the angle described $+BAC$, and $-BAC$ will denote the angle described when B revolves the same distance in the *opposite* direction.

It is thus plain that the *algebraic*, or *symbolic*, *sum* of two quantities may be either *positive* or *negative*. For in the statement, a man walked 6 yds. + 7 yds., if he walked six yards *eastward* and then seven yards *westward*, his position will be represented by (+ 6 yds.) + (− 7 yds.) = − 1 yd.; *i. e.*, he will be one yard to the west of the starting point. But if the man walks 6 yds. west and then 7 yds. east, his position will be represented by (− 6 yds.) + (+ 7 yds.) = + 1 yd., *i. e.*, one yard to the east of the starting point. In both cases the *arithmetical sum* is 6 yds. + 7 yds. = 13 yds., the total distance walked, without regard to direction.

Thus we see that the algebraic sign of such a quantity as $(a + b)$ can not be determined unless we know both the magnitudes and the affections of a and b. If a be positive and b negative, and $a > b$, then $(a + b)$ is positive; but, if $a < b$, then $(a + b)$ is negative; thus if $a = 6$, $b = -9$, $(a + b) = -3$; but if $a = 16$, $b = -9$, $(a + b) = +7$. Similar observations hold good with regard to the *arithmetical* and the *algebraic difference* of two quantities.

This *opposition* (or *contrariety*) serves to reconcile some apparent anomalies which arise from the more general meaning of algebraic symbols as compared with those used in pure Arithmetic, *e. g.* a^{+2} means $1 \times a \times a$, what does a^{-2} mean? As $+^2$ denotes here *multiplication by two factors*, $-^2$ will mean *division by two factors; i. e.* $1 \div (a \times a)$, or $\frac{1}{a^2}$. Similarly $a^1 = 1 \times a$, $a^{-1} = 1 \div a$. Thus, $a^1 \times a^{-1} = a^0$, *i. e.*, $(1 \times a) \times \frac{1}{a} = 1$, so that $a^0 = 1$. And, so generally, as $a^n = 1 \times a \times a \times a$ to n factors, $a^{-n} = \frac{1}{a \times a \times a \text{ to } n \text{ factors}}$.

In the preceding fundamental rules, the signs + and − have been used in their arithmetical meanings. But all the operations considered are true whether the quantities concerned are essentially *positive* or *negative*. Thus in the proof given for the statement that $a - (b - c) = a + b - c$, b was assumed to be greater than c; but it is true if $b < c$; *i. e.*, if $b - c$ is negative. Suppose it is required to subtract: i. $+b$ from a, and ii. $-b$ from a where a and b represent any quantities, $-b$ and $+b$ of course denoting the *same number of units*. Then by the definition of the signs + and −, $0 = +b - b$. Now $a = a + 0 = a + b - b$; i. from this take away $+b$, and $-b$ is the result, and ii. take away $-b$ and $+b$ is the result.

CHAPTER VI.

SIMPLE EQUATIONS.—PROBLEMS.

70. An Equation is a statement, in the language of Algebra, that two expressions are equal.

The two expressions are called *members*, or *sides*, of the equation, the one to the left of the sign of equality, the *left member*, or side, the one to the right of the sign of equality, the *right member*, or side.

Thus, $2x - 7 = 3 + x$, is an equation; $2x - 7$ is the *left side*, and $3 + x$, the *right side*.

71. If an equation involves arbitrary numbers (Art. 9), that is, if it is true for *all values* of the symbols it contains, it is called an *Identical Equation*, or simply an *Identity*; thus,

$2x + 4 = x + x + 4$, and

$(x + a)(x - a) = x^2 - a^2$ are *identities*, for they are true for *all*, or any, *values* that may be assigned to the letters.

72. If an equation is true only for some particular value, or values, of the *unknown* quantities (Art. 8), it is called an *equation of condition*, or simply an **Equation**. To this meaning, the word is generally restricted.

Thus, the equation $2x - 7 = 3 + x$ is true for $x = 5$, and for *no other finite value of x*.

73. Any value of the unknown quantity for which an equation is true is called a **root** of the equation.

74. The process of finding any such root is called *solving* the equation. When a root is substituted for the unknown quantity in an equation, the equation becomes an *identity*, and the root is said to be *verified*, and the equation, *satisfied*.

75. A simple equation is one which contains only the *first power* of the unknown quantity; thus,

$6x = 24$, $ax + b = c$, are simple equations. An equation in which the highest power of the unknown quantity is the *second*, is called an *equation of the second degree*, or a **Quadratic Equation**, thus $x^2 + a = 0$, $x^2 + ax + b = 0$ are *Quadratic* equations.

76. The following AXIOMS, which are obviously true in Arithmetic, are true also in Algebra, and are of use in solving equations :—

1. Things which are equal to the same thing are equal to one another.
2. If equals be added to or subtracted from equals, the results are equal.
3. If equals be multiplied or divided by equals the results are equal.
4. The same powers of equals are equal.
5. The same roots of equals are equal.
6. A quantity is not altered in value by the addition of a pair of *complementary terms* (Art. 18).

77. Solution of Simple Equations.—A simple equation may be solved by reducing it to the form $ax = b$, and dividing both members by a.

78. Transposition.—If the equation has no fractions, it is reduced to this form simply by the *transposition* of its terms, which consists in removing all the terms involving the unknown quantity to one side of the equation, and all the remaining terms to the other side. Any term may be *transposed* by simply *changing* its *sign* (Art. 76. Ax. 2).

79. Clearing of Fractions.—If the equation contains fractions, it must be *cleared of fractions*. This is done by multiplying every term of both sides of the equation by *the least common multiple of the denominators of the fractions* (Art. 76. Ax. 3).

Ex. 1 . $$6x + 18 = 24$$

subtract 18 from both sides $$\therefore 6x + 18 - 18 = 24 - 18,$$

or $$6x = 24 - 18, \qquad (1)$$

or $$6x = 6, \quad \therefore x = 1. \qquad (2)$$

We see that equation (1), which we obtained by *taking equals from equals*, may be got at once by transferring 18 to the right side, and changing its sign.

Ex. 2. $ax + b = cx + d;$

Add $-b$ to both sides $\therefore \quad ax = cx + d - b;$

Add $-cx$ " " " $\therefore \quad ax - cx = d - b$

$\therefore \quad (a - c)\,x = d - b$

$\therefore \quad x = (d - b) \div (a - c).$

Ex. 3. $$\frac{2x}{3} + \frac{x}{2} - 1\tfrac{5}{12} = 18\tfrac{5}{12} - \frac{x}{4}:$$

Here we must clear of fractions.

The l. c. m. of the denominators is 12; multiply both sides by 12.

$\therefore \quad 8x + 6x - 17 = 221 - 3x;$

now transpose $\therefore \quad 8x + 6x + 3x = 221 + 17$

$\therefore \quad 17x = 238$

$\therefore \quad x = 14.$

Ex. 4. $$3x - \left\{\tfrac{1}{2}x + 11 - \left(\frac{x}{3} - 4\right)\right\} = 5;$$

remove brackets $\therefore \quad 3x - \tfrac{1}{2}x - 11 + \dfrac{x}{3} - 4 = 5;$

clear of fractions $\therefore \quad 18x - 3x - 66 + 2x - 24 = 30$

transpose $\therefore \quad 18x - 3x + 2x = 30 + 66 + 24$

collect terms $\therefore \quad 17x = 120$

$\therefore \quad x = 7\tfrac{1}{17}.$

N. B.—WHATEVER OPERATION IS PERFORMED ON ONE SIDE OF AN EQUATION MUST BE PERFORMED ON THE OTHER; otherwise the equality between its members will be destroyed.

EXERCISE XXX.

Solve the equations :—

1. $500 - 32x = 12 + 30x - 8.$ 8

2. $2x + \frac{1}{2} = 3x - \frac{1}{2}.$ 1

3. $5\,(\overset{x}{w} - 3) - 7\,(6 - \overset{x}{w}) + 3 = 24 - 3\,(8 - x).$ [illegible] 6

4. $8\,(x - 1) + 17\,(x - 3) = 4\,(4x - 9) + 4.$ 3

5. $4\,(x + 5)^2 - 180 = (2x + 1)^2 + 3\,(x - 5).$ 2

6. $(x+1)(x+2)(x+3)-(x+4)(x+5)(x-3)=84.$ x

7. $(x+1)(x+2)(x+6)-9x^2=x^3+4(7x-1).$ x = 2

8. $7x-5[x-\{7-6x+18\}]=3x+1.$ x = 4

9. $.375x-1.875=.12x+1.185.$ x = 12

10. $a(x-a)+b(x-b)+x^2=(x-a)(x-b).$ $\frac{a^2 + [illegible] ab}{[illegible])}$

11. $ax+b=mx+n+(bx-a)^2-b^2x^2-cx.$ x = [illegible]

12. $ab+ax=bc-cx-b^2x-(c-a)^2x.$

13. $(x-a)(x-b)=(x-a-b)^2.$

14. $x^2-ax=x^2-5x+c+\frac{1}{2}(a-b)x-\frac{1}{3}(c-a)x.$ $\frac{6c}{30 - [illegible]a + 3b + [illegible]c}$

15. $(x-a)^3(x+a-2b)=(x-b)^3(x-2a+b).$ $\frac{1}{2}(a+b)$

16. $(x+a)(2x+b+c)^2=(x+b)(2x+a+c)^2.$

17. $\frac{1}{3}(x-4)+\frac{1}{35}(2x-3)=\frac{1}{5}(5x-32)-\frac{1}{28}(x+9).$ 19

18. $\frac{7x}{8}-\frac{3x}{7}+1\frac{1}{8}=\frac{9x}{4}+\frac{9x}{14}-20\frac{25}{28}.$ 9

19. $\frac{1}{8}(7x+9)-\frac{1}{7}(3x+1)=\frac{1}{4}(9x-13)-\frac{1}{14}(249-9x).$ 9

20. $\frac{1}{29}(2x+1)-\frac{1}{12}(402-3x)=9-\frac{1}{2}(471-6x).$ 72

21. $\frac{1}{7}\left(x-\frac{1}{2}\right)-\frac{1}{5}\left(\frac{2}{3}-x\right)=1\frac{13}{30}.$

22. $3\frac{1}{2}\cdot\left\{2\frac{1}{3}+\frac{x}{4}\right\}=3\frac{1}{3}\cdot\left\{28-\left(\frac{x}{8}+24\right)\right\}.$ 4

23. $\frac{7x-8}{13}+\frac{3x-2}{7}=\frac{10x+3}{11}-\frac{7x-3}{18}.$ 3

24. $x-\frac{a^2-3bx}{a^2}-b^2=\frac{bx}{a}+\frac{6bx-5a^2}{2a^2}-\frac{6x+4a}{4a}.$ $\frac{2ab^2-5a}{2a-2b+3}$

25. $\frac{3-.125x}{8}+.1\dot{6}=\frac{2+.1875x}{7}-.08\dot{3}.$ 8

$\frac{4a[illegible] - 10a^2}{4a^2 - 4ab + 6a}$

80. Solution of Problems.—When a problem is proposed for solution the unknown number is related to certain given numbers; these relations, expressed in the language of Algebra, form an equation, by the solution of which the unknown number is found. (See Chap. III.)

PROBLEMS.

Ex. 1. Find a number such that if $3\frac{3}{4}$ be added to the double of it, five times the sum is equal to four times the number diminished by 3, and the remainder divided by 6.

Let $x =$ number required

then, $(2x + 3\frac{3}{4})\,5 = \dfrac{4x - 3}{6}.$

Multiply through by 12, *i. e.*, L.C.M. of 4 and 6.

$$(2x + 3\tfrac{3}{4})\,60 = 8x - 6\,;$$
$$\therefore\ 120x + 225 = 8x - 6.$$

Transpose like terms,

$$\therefore\ 120x - 8x = -6 - 225\,;$$
$$i.\ e.,\ 112x = -231,$$

or $x = -\frac{231}{112}.$

In this case the *root* is *fractional* and *negative.*

Ex. 2. Find a number such that if k be added to the double of it, five times the sum is equal to four times the number diminished by w, and the remainder divided by $6m$.

Let $x =$ the required number;

then, $(2x + k)\,5 = \dfrac{4x - w}{6m}.$

Multiply through by $6m$,

$$\therefore\ (2x + k)\,30m = 4x - w = 60mx + 30mk.$$

Transpose,

$$\therefore\ 4x - 60mx = w + 30mk = x\,(4 - 60m),$$

where the coefficient of x is $4 - 60m$,

$$\therefore\ x = \frac{w + 30mk}{4 - 60m}.$$

Ex. 3. A smuggler expected to sell a cargo of contraband brandy for \$99; but after he had sold 10 gallons, the officers seized one-third of the remainder, and consequently he received only \$81 for the lot. Find the number of gallons he had, and the price per gallon.

Let $x =$ no. of gallons,

$$\therefore \frac{\$99}{x} = \text{value of one gallon,}$$

and $(x - 10)\frac{1}{3} =$ no. of gallons seized,

and this was worth \$99 − \$81 = \$18.

Thus we find $\frac{1}{3}(x - 10)$ gallons @ $\frac{\$99}{x}$ to be worth \$18; that is,

$$\tfrac{1}{3}(x - 10)\frac{99}{x} = 18.$$

Multiply through by $3x$,

$$(x - 10)\,99 = 54x.$$

Divide through by 9, and

$$11x - 110 = 6x.$$

Transpose, and $5x = 110$,

$$\therefore \quad x = 22 \text{ gallons};$$

and $\therefore$ price $= \frac{99}{22} = \frac{9}{2} = \$4.50$ per gallon.

EXERCISE XXXI.

1. A boy was told to divide half of a certain number by 4, and the other half by 6, and take the sum of the quotients. He attempted to do this by dividing the whole number by 5, but his result came out wrong by 2. What was the number? 240

2. A boy started at 4 miles an hour on an errand to a place 8 miles distant. He had been gone 10 minutes when a messenger was sent at $4\frac{1}{2}$ miles an hour to bring him back. How far had the boy walked when he got back?

3. A student out for exercise walks a certain distance at 4 miles an hour; he then turns back and runs at 10 miles an hour,

arriving at home again in an hour and a half from the time he started. How far did he travel altogether?

4. At a school pic-nic there were three times as many women as men; but when four men and their wives had left, there remained four times as many women as men. How many men were there left at the pic-nic? 8

5. From the nth part of my money take \$$a$ and a times the remainder is equal to \$$b$; how much money have I?

6. After selling a acres more than the mth part of my farm, I have b acres less than the nth part left. Find the number of acres in the farm. $\frac{mn(b+a)}{m+n-mn}$

7. A farmer bought x sheep for \$$y$ and sold z of them at a gain of 5%; at what price must he sell each of the remainder so as to clear 10% on the whole?

8. A mixture contains 70% of wine and the rest is water. What quantity of wine must be added to 100 gallons of the mixture to make it 80% strong? 50

9. A cask is filled with 3 parts alcohol and 1 part water. Part of the mixture is drawn off and the cask is filled up again with water; the mixture is then 50% strong. Find what fraction of the original mixture was drawn off. $\frac{1}{3}$

10. If a cubic inch of gold weighs 20 oz., and a cubic inch of silver 12 oz., and a lump of gold and silver alloy weighs 32 oz. less than it would if it were all gold, and 56 oz. more than if it were all silver, find the actual weight of the lump.

11. A person bought eggs at 36 cents a dozen; had he received 6 more for the same money, they would have cost him 6 cents a dozen less. How many did he buy?

12. If a certain number be multiplied by 12, and this product be diminished by 36, $\frac{3}{8}$ of the remainder will be 4 more than the original number. Find the number. 4

13. A workman was hired for 60 days; for every day he worked he received $1.50, and for every day he was idle he forfeited 50 cents; at the end of the time he received $50. How many days did he work? 40

14. Sold goods at a gain of 10 per cent; had they cost me $20.20 more, the same selling price would have resulted in a loss of $12\frac{1}{2}$ per cent. Find the cost of the goods. [illegible]

15. The length of a field is twice its breadth; another field which is 50 yards longer and 10 yards broader contains 6800 square yards more than the former. Find the size of each. 162 [illegible] 2200 sq. yds.

✓ 16. A number consists of two digits, the units' figure being 3 less than the tens' figure; the number formed by inverting the digits is 27 less than the original number. Find it. 30, 41, 52, 63, [illegible]

17. Find how much water must be mixed with 80 gallons of spirits which cost 15 shillings a gallon, so that by selling the mixture at 12 shillings a gallon there may be a gain of 10 per cent. 30 gal

18. A walks $3\frac{3}{4}$ miles an hour, and starts $18\frac{1}{2}$ minutes before B; find B's rate if he overtakes A at the ninth milestone. 4 [illegible]

19. Two persons, A and B, could do a work in *a* days; they worked together *b* days, when A was called off and B finished it in *c* days. In what time could each do the work? A, $\frac{bc}{b+c-a}$ days; B, [illegible]

20. A grocer expected to realize £9 18*s.* on a cask of syrup; after selling 10 gallons, he lost a third of the remainder by leakage, and so he received altogether only £8 2*s.* Find the number of gallons he had, and the price per gallon.
22 gal. @ [illegible]

21. At a certain railway station $60 was taken for single fares, and $135 for returns. The number of return tickets exceeded the number of single tickets by 10, and the price of a return ticket was 50 per cent more than the price of a single ticket. Find the price of a single ticket.

22. Find three consecutive numbers such that a third of the greatest may be 2 less than a fifth of the other two together. 37, 38, 39

23. Find three numbers in the proportion of 2, 3, 6, such that when each is diminished by 4, the products of the first and third and of the first and second may be, together, 32 less than the product of the second and third. 6, 9, 18

✓ 24. A number has three digits, the units' figure being 2 more than the tens' figure, and 3 more than the hundreds' figure. If the order of the digits were inverted and 62 subtracted, the result would be double the original number. Find it. 235

25. A person going at the rate of a miles an hour, finds himself b hours behind time when he has yet c miles to go. How much must he increase his speed to reach home in time? $\frac{a^2b}{c-ab}$

26. There are three consecutive numbers whose product is equal to 21 times their sum. Find them. 7, 8, 9

27. A and B can do a piece of work in p days, B and C in q days, and C and A in r days; in what time would all together do the work? $\frac{2pqr}{pq+qr+rp}$

28. A man owes a debt of c dollars; he has bank notes of two different values; it would take a notes of the first kind to pay the debt and ma notes of the second kind; he uses some of both kinds and pays the debt with b notes. How many of each kind did he use?

✓ 29. A number consists of three digits which are consecutive numbers; if 198 be subtracted the remainder will be the number formed by inverting the digits of the given number. Find the number.

30. A man bought oranges at m cents a dozen; had he received n more for the same money, they would have cost him p cents a dozen less; how many did he buy?

31. Divide 100 into four parts such that if the first be increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, the results shall all be equal.

CHAPTER VII.

SPECIAL FORMS OF MULTIPLICATION.—INVOLUTION.—EVOLUTION.

81. Certain results (*formulas*) in Multiplication are of great use in shortening labour. The student must therefore memorise these results, and acquire ease and rapidity in applying them.

82. We obtain by actual multiplication :—

$$(x + a)^2 = x^2 + 2ax + a^2 \qquad \text{(A)}$$

$$(x + a)(x - a) = x^2 - a^2 \qquad \text{(B)}$$

In these results the letters represent **any algebraic quantities whatever.** Thus in (A), write $-a$ for a, and we get $(x - a)^2 = x^2 - 2ax + a^2$. In (B), write $(a + b)$ for a and $(c + d)$ for x, and we have :—

$$[(c+d)+(a+b)][(c+d) - (a + b)] = (c + d)^2 - (a + b)^2$$
$$= c^2 + d^2 + 2cd - a^2 - b^2 - 2ab.$$

Ex. 1. $(x + 7)^2 = x^2 + 2(x)\cdot 7 + 7^2 = x^2 + 14x + 49$, by (A).

Ex. 2. $(2x - 11)^2 = (2x)^2 - 2(2x)\cdot 11 + 11^2 = 4x^2 - 44x + 121$, by (A).

Ex. 3. $(\frac{7}{11}a^2 - \frac{5}{7}b^2)^2 = (\frac{7}{11}a^2)^2 - 2(\frac{7}{11}a^2)(\frac{5}{7}b^2) + (\frac{5}{7}b^2)^2$,
$= \frac{49}{121}a^4 - \frac{10}{11}a^2b^2 + \frac{25}{49}b^4$, by (A).

Ex. 4. $(x + 6)(x - 6) = x^2 - 6^2 = x^2 - 36$, by (B).

Ex. 5. $(7x + 11)(7x - 11) = (7x)^2 - 11^2 = 49x^2 - 121$, by (B).

Ex. 6. $(a^2+ab+b^2)(a^2-ab+b^2) = [(a^2+b^2) + ab][(a^2 + b^2) - ab]$
$= (a^2 + b^2)^2 - (ab)^2 = a^4 + 2a^2b^2 + b^4 - a^2b^2 = a^4 + a^2b^2 + b^4$.

Ex. 7. $(w + x - y)(w - x + y) = [w + (x - y)][w - (x - y)]$
$= w^2 - (x - y)^2 = w^2 - x^2 - y^2 + 2xy$.

Ex. 8. $(a + b - c - d)(a - b + c - d) = \{(a - d) + (b - c)\} \times$
$\{(a - d) - (b - c)\} = (a - d)^2 - (b - c)^2$.

83. These formulas may be expressed in words thus :—

A.—*The square of any binomial is equal to the square of each term and twice the product of the two terms.*

B.—*The sum of two quantities into their difference is equal to the difference between the squares of the two quantities.*

EXERCISE XXXII.

(*a*)

Obtain the squares of the following quantities :—

1. $x + a$; $x + 1$; $x + 3$; $x + 4$; $x + b$; $x + \frac{1}{2}$.

2. $x - a$; $x - 2$; $x - 3$; $x - 4$; $x - b$; $x - \frac{1}{2}$.

3. $1 + x$; $1 - x$; $2 + x$; $2 - x$; $3 + 2x$; $3 - 2x$.

4. $x - 12$; $x + 10$; $2x - 1$; $2x + 3$; $3x - 1$; $4x - 3$.

5. $2x - y$; $2x - 3y$; $1 + 3x$; $2 - 3x$; $3 + 4x$; $x^2 - y^2$.

6. $\frac{1}{2} + x$; $\frac{1}{2} - x$; $\frac{1}{2} + 2x$; $\frac{1}{2} - 3x$; $3x - \frac{1}{3}$; $2x - 3y$.

7. $x - \frac{3}{4}$; $2ax - 3y$; $ax - 1$; $ax - y$; $1 - by$; $by + 2$.

8. $\frac{1}{2}x^2 - y^2$; $ax^2 + y^2$; $by^2 + x^2$; $ax - by$; $5x - 1$; $1 - \frac{1}{3}y^2$.

9. $7x + \frac{1}{7}$; $\frac{1}{2}x + \frac{1}{2}y$; $\frac{1}{4}x^2 - 4y^2$; $(a + b) + c$; $(x - y) + 3$.

Obtain the products of the following quantities :—

10. $x - 1$, $x + 1$; $x - 3$, $x + 3$; $2x + 4$, $2x - 4$; $3x + 4$, $3x - 4$.

11. $a + 2b$, $a - 2b$; $1 + x$, $1 - x$; $2x + y$, $2x - y$.

12. $x - 3$, $x + 3$; $x - 7$, $x + 7$; $x - 10$, $x + 10$; $1 + 7x$, $1 - 7x$.

13. $2x + 3$, $2x - 3$; $4x - b$, $4x + b$; $\frac{1}{2} + 4b$, $\frac{1}{2} - 4b$;

14. $x - 11a$, $x + 11a$; $3x + \frac{1}{3}$, $3x - \frac{1}{3}$; $1 - \frac{1}{3}x$, $1 + \frac{1}{3}x$.

15. $2x - \frac{1}{4}$, $2x + \frac{1}{4}$; $ab - 1$, $ab + 1$; $1 - a^2b^2$, $1 + a^2b^2$.

16. $3x^2 + y^2,\ 3x^2 - y^2$; $1 + y^3,\ 1 - y^3$; $2a^4 - b^4,\ 2a^4 + b^4$.

17. $5x^2 - \frac{1}{10},\ 5x^2 + \frac{1}{10}$; $1 - 2abc,\ 1 + 2abc$; $x^5 + y^5,\ x^5 - y^5$.

18. $a^2 + b,\ a^2 - b$; $b^5 - a^3,\ b^5 + a^3$; $b^5 + 1,\ b^5 - 1$.

19. $a^4 - b^2,\ a^4 + b^2$; $x + \frac{1}{x},\ x - \frac{1}{x}$; $a^m + b^m,\ a^m - b^m$.

20. $a^6 - b^3,\ a^6 + b^3$; $1 + ax^n,\ 1 - ax^n$; $ax^{2n} + by^{2n},\ ax^{2n} - by^{2n}$.

21. $(a + b) + c,\ (a + b) - c$; $(x + y)^2 + z^2,\ (x + y)^2 - z^2$.

22. $(1 + x)^3 + y^3,\ (1 + x)^3 - y^3$; $(2 + x)^4 - y^2,\ (2 + x)^4 + y^2$.

(*b*)

Write down the squares of

1. $13x - 2a$; $15x - \frac{1}{2}a$; $21xy + 3x$; $12ab^2 - 6a^2bc$.

2. $\frac{2}{3}x^2 + \frac{3}{4}y^2$; $4\frac{1}{4}a^2b - \frac{4}{17}ab^2$; $17x^2y^3z^4 - \frac{1}{17}x^4y^3z^2$.

3. $1000 + 12$; $1000 - 2$; $100 + 11$; $70 + 5$; $50 - 1$.

4. $5a^{58}b^7 + 6a^{91}b^5$; $a^{112} - b^{20}$; $77a^{77} + 88b^{88}$.

Find the products of

5. $(13x+2)(13x-2)$; $\left(\frac{1}{2}x+\frac{1}{20}\right)\left(\frac{1}{2}x-\frac{1}{20}\right)$; $(2x^2-\frac{1}{12}y^2)(2x^2+\frac{1}{12}y^2)$.

6. $(7x^7 - 16xy)(7x^7 + 16xy)$; $(\frac{1}{7}x^7 + \frac{1}{6}xy)(\frac{1}{7}x^7 - \frac{1}{6}xy)$;
$(x^{77} + y^{88})(x^{77} - y^{88})$.

7. $(1000+12)(1000-12)$; $(100+11)(100-11)$; $(70+5)(70-5)$;
$(50 + 1)(50 - 1)$.

8. $(a^{112} - b^{200})(a^{112} + b^{200})$; $(5a^{58}b^{72} - 6a^{91}b^{57})(5a^{58}b^{72} + 6a^{91}b^{57})$;
$(77a^{77} + 88b^{88})(77a^{77} - 88b^{88})$.

Simplify the following expressions :—

9. $3(a - 2x)^2 + 2(a - 2x)(a+2x) + (3x - a)(a+3x) - (2a - 3x)^2$.

10. $(a + b + c)(a + b - c)$; $(x - y + z)(x - y - z)$.

$a^2+2ab+b^2-c^2$

11. $(2a + b - 3c)(2a - b + 3c)$; $(2x + y - 3z)(y + 3z - 2x)$.

$4a^2-b^2+6bc-9c^2$

12. $(w + x + y + z)(w - x + y - z)$; $(r+s+t+u)(s+t-u-r)$.

13. $(a - 2b + 3c + d)(a - 3c + 2b + d)$;

a^2+2a [illegible] $-4a^2+12bc-9c^2$ $(2k - x + 3y + z)(x - 2k + 3y + z)$.

14. $(m + 2p + 3s - k)(m - 2p + k + 3s) -$
$(m + 3s - k + 2p)(2p - m + 3s + k)$.

$2m^2 +$ [illegible] -12

15. Show that $(a+b)^2+(b+c)^2+(c+a)^2-(a^2+b^2+c^2)$
$$= 3(a^2+b^2+c^2)-\{(a-b)^2+(b-c)^2+(c-a)^2\}$$
$$= \tfrac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2+6(ab+bc+ca)\}.$$

84. Resume formula (A),

$(x+a)^2=x^2+2ax+a^2$. Write $a+b$ for a, and we have

$$\begin{aligned}(x+a+b)^2&=x^2+2x(a+b)+(a+b)^2\\&=x^2+a^2+b^2+2ax+2bx+2ab. \qquad \text{(i.)}\end{aligned}$$

Again, in (i.) write $c+d$ for x and we get

$$\begin{aligned}&(a+b+c+d)^2\\&=(a+b)^2+(c+d)^2+2(a+b)(c+d)\\&=a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd. \qquad \text{(ii.)}\end{aligned}$$

These results may also be obtained by actual multiplication. If in (ii) we write $d+e$ for d, we get $(a+b+c+d+e)^2$, and it will be found to resemble (A) and (i) and (ii) *in form*, *i. e.* the expansion will contain only *two kinds of terms*, viz. *perfect squares* like x^2, a^2, b^2, c^2, etc. and *doubled products*, like $2ax$, $2bx$, $2bd$, etc. We are thus enabled to generalise (A), (i), (ii) into one formula expressing the square of any algebraic polynomial :—

$(a+b+c+\ldots.)^2=(a^2+b^2+c^2+$, etc.$)+(2ab+2ac+$, etc., $+\,2bc+2bd+$ etc., $+\,2cd+$ etc., all possible double products). (C).

If the terms contain different letters, it is convenient to write the squares first as shown above, and then form the doubled products; thus :—

$2a(b+c+$ etc. all the following terms$)+2b(c+d+$ etc. all the following terms$)+$ etc.

Ex. 1. $(2a+3b+4c)^2=(2a)^2+(3b)^2+(4c)^2+4a(3b+4c)+6b(4c)$
$$=4a^2+9b^2+16c^2+12ab+16ac+24bc.$$

Ex. 2. $(\frac{1}{2}a+\frac{1}{4}b+\frac{1}{6}c)^2=\frac{1}{4}a^2+\frac{1}{16}b^2+\frac{1}{36}c^2+a(\frac{1}{4}b+\frac{1}{6}c)+\frac{1}{2}b\cdot\frac{1}{6}c$
$$=\tfrac{1}{4}a^2+\tfrac{1}{16}b^2+\tfrac{1}{36}c^2+\tfrac{1}{4}ab+\tfrac{1}{6}ac+\tfrac{1}{12}bc.$$

Ex. 3. $(a-2b+3c-4d)^2=(a)^2+(-2b)^2+(3c)^2+(-4d)^2$
$$2a(-2b+3c-4d)-4b(3c-4d)+6c(-4d)$$
$$=a^2+4b^2+9c^2+16d^2-4ab+6ac-8ad-12bc+16bd-24cd.$$

Ex. 4. $(1-2x+3x^2-4x^3)^2=1-4x+6x^2-8x^3$
$$\begin{array}{r}+4x^2-12x^3+16x^4\qquad\qquad\\+9x^4-24x^5+16x^6\\=1-4x+10x^2-20x^3+25x^4-24x^5+16x^6.\end{array}$$

85. The formula (C) may thus be expressed in words :—

C.—*The square of a polynomial is equal to the* **square of each term**, *together with* **twice each term** *into all the terms that follow it.*

EXERCISE XXXIII.

(*a*)

Give the squares of the following :—

1. $a + b + c$; $a + b - c$; $a - b + c$; $-a + b + c$.
2. $1 + b + c$; $1 + b - c$; $1 - b + c$; $-1 + b + c$.
3. $a - b - c$; $1 - b - c$; $1 + a + 2b$; $1 - a + 2b$.
4. $2 + x - y$; $3 - x + y$; $4 - y + x$; $1 + 2x - y$.
5. $x + 2y + 3z$; $x - 2y + 3z$; $2y - x + 3z$; $1 - 2x - 2y$.
6. $1 + x + x^2$; $1 - x + x^2$; $1 + 2x + x^2$: $1 - 2x + x^2$.
7. $4 + x - 2y$; $5 - y - 3z$; $1 - x - x^2$; $x^2 + y^2 + z^2$.
8. $1 + x^2 + 3x^3$; $1 - x^2 + 3x^3$; $2 - y + 2y^2$; $2x^2 + y - 1$.
9. $1 - x + 2x^2$; $1 + x - 3x^2$; $2a^2 - a - 2$; $1 + a^2 + a^3$.
10. $1 + x + by$; $1 + ax + by$; $1 - ax - by$; $1 - ax^2 + bx^3$.
11. $1 + x + x^2 + x^3$; $1 - 3x + 3x^2 - x^3$; $1 - x - x^2 - x^3$.
12. $1 - 2ax + 3a^2x^2$; $x^3 - 3x^2 + 2x - 1$: $x^3 - 2x^2 + 3x + 4$.
13. $2a - b + 2c$; $a - \frac{1}{2}b + \frac{1}{2}c$; $\frac{1}{2}a - \frac{1}{2}b + c$; $\frac{1}{2}a - b + \frac{1}{3}c$.

(*b*)

1. One side of a square field is $(2x + y)$ rods long, express the area of the field in acres.
2. Find the price in dollars of $(a + b - c)$ articles at $(a + b - c)$ cents apiece.
3. In a certain tract of land there are $(x + 2y + 3w + 4z)$ townships, and each township is divided into $(3w + 2y + x + 4z)$ squares; find the total number of squares.
4. Three fishers went sailing out into the West; each fisher left $ab + bc - ca$ children at home, and brought back for each child twice as many fish as there were children in the family. Express the total haul of fish brought home.

5. Show that $(ab + bc + ca)^2 = a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a+b+c)$.

6. Simplify $(w + x + y + z)^2 + (w - x - y + z)^2 + (w - x + y - z)^2$
$+ (w + x - y - z)^2$.

7. Simplify $(x^2 + xy + y^2)^2 - (x^2 - xy + y^2)^2 + (x^2 + xy - y^2)^2$
$- (x^2 - xy - y^2)^2$.

8. Find without actual multiplication the product of
$a^4 - 2a^3 + 3a^2 - 2a + 1$ and $a^4 + 2a^3 + 3a^2 + 2a + 1$.
$a^8 + 2a^6 + 3a^4 + 2a^2 + 1$

9. Show that $(aw + bx + cy + dz)^2 + (ax - bw + cz - dy)^2$
$+ (ay - cw - bz + dx)^2 + (az - dw + by - cx)^2$
$= (a^2 + b^2 + c^2 + d^2)(w^2 + x^2 + y^2 + z^2)$.

10. Find without actual multiplication the continued product of
$(a + b + c)(b + c - a)(c + a - b)(a + b - c)$.
$2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4$

86. The formulas (A) and (B) are particular cases of the following more general result, which may be proved by multiplication:—

$$(x + a)(x + b) = x^2 + x(a + b) + ab. \qquad \text{(D)}$$

If we write $+a$ for b in this result, we get formula (A); if we write $-a$ for b we get formula (B).

Ex. 1. $(x + 1)(x + 2) = x^2 + x(1 + 2) + 2.1 = x^2 + 3x + 2$.

Ex. 2. $(3x + 1)(3x + 2) = (3x)^2 + 3x(1 + 2) + 2.1 = 9x^2 + 9x + 2$.

Ex. 3. $(x + a)(x - b) = x^2 + x(a - b) - ab$.

Ex. 4. $(x - a)(x - b) = x^2 + x(-a - b) + ab = x^2 - x(a + b) + ab$.

Ex. 5. $(x + a + b)(x + a - 3b)$
$= (x + a)^2 + (x + a)(b - 3b) + (b)(-3b)$
$= x^2 + 2ax + a^2 - 2bx - 2ab - 3b^2$.

Ex. 6. $(\overline{x - y + z} - w)(\overline{x - y + z} + 3w)$
$= (x - y + z)^2 + (x - y + z)\cdot 2w - 3w^2 =$ etc.

Ex. 7. $(mx + a)(mx + b) = m^2x^2 + (a + b)mx + ab$.

This formula may be expressed in words thus:—

D.—The product of two binomials which have the first terms alike is equal to *the algebraic sum of* 1°, *the square of the common term*; 2°, *the product of the sum of the unlike terms into the common term*; and 3°, *the product of the unlike terms.*

87. By actual multiplication we get, with another factor :—

$$(x+a)(x+b)(x+c) = x^3 + x^2(a+b+c) + x(ab+bc+ca) + abc. \quad \text{(E)}$$

Thus we may write down *at sight* the product of three binomial factors whose *first terms are the same*, but the *last terms different.*

Ex. 1 . $(x + 1)(x + 2)(x + 3)$
$= x^3 + x^2(1 + 2 + 3) + x(1\cdot2 + 2\cdot3 + 3\cdot1) + 1\cdot2\cdot3$
$= x^3 + x^2(1 + 2 + 3) + x(2 + 6 + 3) + 6$
$= x^3 + 6x^2 + 11x + 6.$

Ex. 2 . $(3x + 4)(3x + 5)(3x + 6)$
$= (3x)^3 + (3x)^2\cdot(4 + 5 + 6) + 3x(20 + 30 + 24) + 120$
$= 27x^3 + 135x^2 + 222x + 120.$

88. Similarly, we may obtain by multiplication :—

$$\begin{aligned}&(x + a)(x + b)(x + c)(x + d)\\ &= x^4 + x^3(a + b + c + d) + x^2(ab + bc + cd + da + ac + bd)\\ &+ x(abc + bcd + cda + dab) + abcd.\end{aligned}$$

Comparing (D), (E), and the last result, we see that the law of such products is very simple :—

1°. *The first term is the product of all the x's, and the last term is the product of all the second terms of the factors, a, b, c, etc.*

2°. *The coefficient of the second term is the sum of a, b, c, etc.*

3°. *The coefficient of the third term is the sum of the products of a, b, c, etc., multiplied two and two together.*

4°. *The coefficient of the next term is formed by taking them three and three together, and so on.*

89. This law is perfectly general, and may be briefly written for n factors :—

$$(x+a_1)(x+a_2)\ldots\ldots(x+a_n) = x^n + c_1x^{n-1} + c_2x^{n-2} + c_3x^{n-3} + \ldots + c_n, \quad \text{(F)}$$

where

$c_1 = a_1 + a_2 +$ etc. $\ldots + a_n$; $c_2 = a_1a_2 + a_2a_3 +$ etc.; $c_3 = a_1a_2a_3 + a_2a_3a_4$ + etc., and $c_n = a_1a_2a_3 \ldots\ a_n.$

90. If $a = b = c = d =$ etc., in formulas (D), (E), or $a_1 = a_2 =$ etc. $= a_n$ in (F), we get

$$(x + a)^2 = x^2 + 2ax + a^2.$$

$$(x + a)^3 = x^3 + x^2(3a) + x(3a^2) + a^3 = x^3 + 3ax^2 + 3a^2x + a^3.$$

$$\begin{aligned}(x + a)^4 &= x^4 + x^3(4a) + x^2(6a^2) + x(4a^3) + a^4 \\ &= x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4.\end{aligned}$$

$$(x + a)^5 = x^5 + 5ax^4 + 10a^2x^3 + 10a^3x^2 + 5a^4x + a^5.$$

$$(x + a)^6 = x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

These results may easily be verified by actual multiplication. A careful study of these products reveals a very simple law of coefficients. Observe,

1°. *The numerical coefficient of any term is found by multiplying the exponent of the letter of reference* (x) *in the preceding term by its numerical coefficient, and dividing by the number of that term from the beginning of the expansion.* Thus in $(x + a)^6$, the coefficient $20 = \frac{4 \times 15}{3}$. And

$$\begin{aligned}(x+a)^7 &= x^7 + \frac{7 \times 1}{1}ax^6 + \frac{6 \times 7}{2}a^2x^5 + \frac{5 \times 21}{3}a^3x^4 + \frac{4 \times 35}{4}a^4x^3 + \text{etc.} \\ &= x^7 + 7ax^6 + 21a^2x^5 + 35a^3x^4 + 35a^4x^3 + 21a^5x^2 + 7a^6x + a^7.\end{aligned}$$

2°. *After the middle term the same coefficients recur, and therefore do not require to be calculated again. The signs of the terms are included.*

Ex. 1. $(x - a)^5 = x^5 - 5ax^4 + 10a^2x^3 - 10a^3x^2 + 5a^4x - a^5.$

Ex. 2. $(2x - 5)^4 = (2x)^4 - 4(2x)^3(5) + 6(2x)^2(5)^2 - 4(2x)(5)^3 + (5)^4$
$= 16x^4 - 160x^3 + 600x^2 - 1000x + 625.$

Ex. 3. $(x - 2y)^{19}$
$= x^{19} - 19x^{18}(2y) + 171x^{17}(2y)^2 - 17 \times 57x^{16}(2y)^3 +$ etc.
$= x^{19} - 38x^{18}y + 684x^{17}y^2 - 7752x^{16}y^3 +$ etc., 16 more terms.

Ex. 4. $(\frac{3}{2}a + \frac{2}{3}b)^3 = (\frac{3}{2}a)^3 + 3(\frac{3}{2}a)^2(\frac{2}{3}b) + 3(\frac{3}{2}a)(\frac{2}{3}b)^2 + (\frac{2}{3}b)^3$
$= \frac{27}{8}a^3 + \frac{9}{2}a^2b + 2ab^2 + \frac{8}{27}b^3.$

EXERCISE XXXIV.

(*a*)

Find the products in the following cases :—

1. $x + 3, x + 4$; $x + 4, x + 5$; $x + 5, x + 6$; $x + 6, x + 7$.
2. $x + 5, x - 4$; $x - 4, x - 5$; $x - 5, x - 6$; $x - 6, x - 7$.
3. $x + 6, x - 4$; $x + 4, x - 3$; $x + 8, x - 6$; $x + 10, x - 9$.
4. $x - 6, x + 3$; $x - 7, x + 5$; $x - 10, x + 4$; $x - 9, x + 8$.
5. $x + 11, x - 12$; $x + 12, x - 11$; $x + 10, x - 9$; $x + 2a, x - a$.
6. $x - 3a, x + 5a$; $x - 9a, x + 10a$; $x - 8a, x + 7a$; $x + 2, x + a$.
7. $x + 1, x + a$; $x + 2a, x + 3$; $x + 3a, x - 4a$; $x + 7a, x - 10a$.
8. $2x + 1, 2x + 3$; $3x - 1, 3x + 7$; $4x - 5, 4x + 7$; $5x - a, 5x + 3a$.
9. $4x + a, 4x + 1$; $3x - a, 3x - 1$; $4x + 2a, 4x + 1$; $a - 3b^2, a - b^2$.
10. $x - \frac{1}{2}, x + 1\frac{1}{2}$; $x^2 + 7, x^2 - 9$; $x - \frac{1}{2}, x - \frac{1}{4}$; $x^2 - \frac{3}{8}, x^2 + 1\frac{1}{4}$.
11. $3a - 1, 3a + b$; $4a^2 + b, 4a^2 - 1$; $3x + 7a, 3x - 7a$; $x + 1, x + b$.
12. $x^2 - a, x^2 - 4$; $x^3 - 2a, x^3 + 5$; $2x^2 - 3a, 2x^2 + 4$.

Expand the following, by Art. 90 :—

13. $(x + y)^3$; $(x - y)^3$; $(1 + x)^3$; $(1 - x)^3$; $(a + 2)^3$; $(a + 3)^3$.
14. $(a + 2b)^3$; $(a - 2b)^3$; $(2a - 1)^3$; $(1 + 2a)^3$; $(2a + 3b)^3$.
15. $(a + b)^4$; $(a - b)^4$; $(1 + a)^4$; $(1 - a)^4$; $(2 + a)^4$; $(2 - 3a)^4$.
16. $(x + 1)^4$; $(1 - x)^4$; $(2 + \frac{1}{2}x)^4$; $(2 - x)^4$; $(1 + 2x)^4$; $(1 - 2x)^4$.
17. $(a + x)^5$; $(a + 2x)^5$; $(1 + x)^5$; $(1 + 2x)^5$; $(1 + x^2)^5$; $(2 + x^2)^5$.

(*b*)

Find the following products, without actual multiplication :—

1. $(a + b)(a + 12b)$; $(a - \frac{1}{2}b)(a + 2b)$; $(a + \frac{3}{4}b)(a - 2b)$; $(a - b)(a - \frac{1}{2}b)$.
2. $(5a + 14b)(5a + 13b)$; $(a^2 + b^2)(a^2 - 12b^2)$; $(2b^2 + a^2)(a^2 - 12b^2)$.
3. $(\frac{2}{3}a - \frac{3}{4}b)(\frac{5}{4}b + \frac{2}{3}a)$; $(\frac{11}{9}abc + 1)(10 + \frac{11}{9}abc)$; $(a^{101} - 5c)(5c + a^{101})$.

4. $(3x + 4y + z)(3x + 4y - 5z)$; $(2a - 3b + 4c)(2a + 3b + 4c)$.

5. $(x^3 + 2x^2 - 3x + 4)(x^3 + 2x^2 - 5x + 4)$;
$(x + y + z + w)(x + 3y + z + w)$.

6. $(x+2)(x+3)(x+4)$; $(x+7)(x+6)(x+1)$; $(x+1)(x+5)(x+3)$.

7. $(x-2)(x-3)(x-4)$; $(x-7)(x-6)(x-1)$; $(x-1)(x-5)(x-3)$.

8. $(x+2)(x-3)(x+4)$; $(x-7)(x-6)(x+1)$; $(x-1)(x+5)(x-3)$.

9. $(2x + 1)(2x + 2)(2x + 3)$; $(2x - 1)(2x - 2)(2x - 3)$;
$(2x - 1)(2x + 2)(2x - 3)$.

10. $(x + y)(x + z)(x + w)(x + k)$; $(x - a)(x - b)(x - c)(x - d)$.

Expand the following, by the method of Art. 90 :—

11. $(w + r)^3$; $(w + r)^4$; $(2w + r)^3$; $(w + 2r)^4$; $(w - r)^3$; $(w-r)^4$.

12. $(k + 3s)^6$; $(a - 2b)^6$; $(2a - \frac{1}{2}w)^3$; $(\frac{1}{2}a + 2w)^3$; $(3a^2 - \frac{1}{3})^3$.

Ans. 13. Find the 4th term of $(a + 2b)^{11}$; $(3a - 2b)^7$; $(5a - \frac{1}{5}x)^5$.

Ans. 14. Write down the last three terms of $(a + b)^{55}$; $(x + y)^{66}$; $(11a - 1)^{11}$.

15. Give the *middle* terms of $(2a + \frac{3}{2}b)^4$; $(\frac{3}{2}a - 2b)^6$; $(a + 2)^6$.

16. At compound interest the amount of \$1 for 5 yrs. at 4% is represented by $(1 + \frac{4}{100})^5$. Expand this expression, express the result decimally, and thus find the *exact* amount of \$1.

17. $(1 + \frac{5}{100})^{11}$ represents the amount of \$1 at compound interest for 11 years @ 5%. Expand this to five terms, and thus find the *approximate* amount of \$1.

91. Special Forms of the Cube.—From what precedes we have

$$\left.\begin{aligned} &(1) & (x \pm a)^3 &= x^3 \pm 3x^2a + 3xa^2 \pm a^3, \\ &(2) & &= x^3 \pm a^3 \pm 3ax(x \pm a). \end{aligned}\right\} \qquad (G)$$

In this substitute $y + z$ for a and we get

$$\left.\begin{aligned} & & &(x + y + z)^3 \\ &(1) & &= x^3+y^3+z^3+3x^2(y+z)+3y^2(z+x)+3z^2(x+y)+6xyz \\ &(2) & &= x^3+y^3+z^3+3xy(x+y)+3yz(y+z)+3zx(z+x)+6xyz \\ &(3) & &= x^3 + y^3 + z^3 + 3(x + y + z)(xy + yz + zx) - 3xyz \\ &(4) & &= x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x) \end{aligned}\right\} \qquad (H)$$

In (H) (1) substitute $w + x$ for x and we get, similarly,

$$(w+x+y+z)^3$$
$$= w^3+x^3+y^3+z^3+3w^2(x+y+z)+3x^2(y+z+w)+3y^2(z+w+x)$$
$$+ 3z^2(w+x+y) + 6(xyz + yzw + zwx + wxy) \qquad \text{(I)}$$

The student should work out all the formulas given, verify them by actual multiplication, and write them out from memory until complete facility in handling them has been acquired.

92. Evolution.—The inverse of the preceding processes is called *extracting a root.* Thus, $(a + b)^2 = a^2 + 2ab + b^2$, $\therefore a + b$ is the *square root* of $a^2 + 2ab + b^2$; $a - b$ is the cube root of $a^3 - b^3 - 3ab(a - b)$, and is the fourth root of $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

Ex. 1. The cube root of a^3 is a; of $-a^3$ is $-a$; of a^3b^6 is ab^2; of $27a^6b^9$, that is, of $3^3\cdot a^6\cdot b^9$, is $3a^2b^3$; of $64x^{12}y^9$, that is, of $4^3\cdot x^{12}\cdot y^9$, is $4x^4y^3$, etc.

The fourth root of a^4 is a; of $81a^8b^{12}$, that is, of $3^4\cdot a^8\cdot b^{12}$, is $3a^2b^3$, etc.

The fifth root of a^5 is a; of $-a^5$ is $-a$; of a^5b^{10} is ab^2; of $32a^{15}b^{10}$, that is, of $2^5a^{15}b^{10}$, is $2a^3b^2$, etc.

The nth root of a^n is a; of a^{3n} is a^3, etc.

It appears, therefore, that **to extract any root of a monomial** we *extract the root of the coefficient, and divide the exponent of each letter by the index of the root, i. e.*, by 2 for the square root, 3 for the cube root, 4 for the fourth root, etc.

Ex. 2. The square root of $9x^2 - 12xy + 4y^2$ is $3x - 2y$; for we see that $9x^2 = (3x)^2$, and $4y^2 = (2y)^2$; they therefore correspond to a^2 and b^2 in $a^2 + 2ab + b^2$. Also $12xy = 2(3x)(2y)$, and the *minus* sign shows that one factor is positive and the other negative; thus, $3x - 2y$ is the square root, as $a - b$ is the square root of $a^2 - 2ab + b^2$.

Ex. 3. $9a^2 - 12ab + 24ac - 16bc + 4b^2 + 16c^2$. To find the square root of this quantity, notice that $9a^2$, $4b^2$, $16c^2$ are the squares of $3a$, $2b$, $4c$ respectively, and that $12ab$, $24ac$, $16bc$ are the doubled products of $(3a)(2b)$, $(3a)(4c)$, and $(2b)(4c)$. We have therefore only to observe the signs of the doubled products to get the proper square root, $3a - 2b + 4c$.

Ex. 4. The cube root of $1 + 6x + 12x^2 + 8x^3$ is $1 + 2x$, for it is $(1)^3 + (2x)^3 + 3\,(1)\,(2x)\,(1 + 2x)$, see (G), (2).

Ex. 5. $8x^6 - 12x^5 + 18x^4 - 13x^3 + 9x^2 - 3x + 1 = (2x^2 - x + 1)^3$; for it is $= (2x^2)^3 - 3\,(2x^2)^2\,(x) +$, etc. $+ (1)^3$. The first and the last terms give us $2x^2$ and 1, and the second term shows that $-x$ is the middle term.

EXERCISE XXXV.

(a)

Find the product of

1. $(x - 5)\,(x + 6)\,(x - 7)\,(x + 8)$.
2. $(x + a)^2 + (x + a)\,(x + b) + (x + b)^2$ and $(x + a)^2 - (x + a)\,(x + b) + (x + b)^2$.
3. $ab + cd + ac + bd$ and $ab + cd - ac - bd$.
4. $a^2 + b^2 + c^2 - bc - ca - ab$ and $a + b + c$. a^3
5. $x^4 - (p - 1)\,x^3 + (q - p + 1)\,x^2 - (p - 1)\,x + 1$ and $x - 1$.
6. $(x^2 + x + 1)\,a - (x + 1)$ and $(x - 1)\,a^2 - (x - 1)\,a + 3$.

Simplify the following expressions :—

7. $(w - 2x)\,(w + 2x) + (2x + 3y)\,(2x - 3y) + (3y - z)\,(3y + z)$.
8. $(x - y)^3 + (x + y)^3 + 3\,\{(x - y)^2\,(x + y) + (x + y)^2\,(x - y)\}$.
9. $(x + y + z)^3 - (x + y - z)^3 - (x - y + z)^3 - (y + z - x)^3$.
10. $(x+y+z)\,(x+y+w) + (x+z+w)\,(y+z+w) - (w+x+y+z)^2$.
11. $(x + y + z)^3 - (x + y)^3 - (y + z)^3 - (z + x)^3 + x^3 + y^3 + z^3$.
12. $(x + y + z)\,(x + y - z)\,(x + z - y)\,(y + z - x)$, when $x^2 + y^2 = z^2$.

$21x^2y^2$

(b)

Show that

13. $(ab + bc + ca)^3 = a^3b^3 + b^3c^3 + c^3a^3 + 3abc\,(a + b + c)\,(ab + bc + ca) - 3a^2b^2c^2$.
14. $(ay - bx)^2 + (cx - az)^2 + (bz - cy)^2 = (a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) - (ax + by + cz)^2$.
15. $(x^2 + y^2 - 1)^2 + (w^2 + k^2 - 1)^2 + 2\,(wx + yk)^2 = (w^2 + x^2 - 1)^2 + (y^2 + k^2 - 1)^2 + 2\,(xy + wk)^2$.

16. $(x-y)^4+(y-z)^4+(z-x)^4=2\,(x^2+y^2+z^2-xy-yz-zx)^2.$

17. $2\,(xy+yz+zx)^3-2x^2y^2z^2$
$=(x+y)^2\,(y+z)^2\,(z+x)^2-x^4\,(y+z)^2-y^4\,(z+x)^2-z^4\,(x+y)^2.$

18. $(a-b)^3+(b-c)^3+(c-a)^3=3\,(a-b)\,(b-c)\,(c-a).$

EXERCISE XXXVI.

(*a*)

Extract the square roots of :—

1. a^2b^2; $4a^2b^2$; $9a^4b^4$; $16\,(a+b)^2$; $36a^4b^4$.

2. $4a^2b^2c^2$; $25a^4b^6$; $25x^4y^4z^2$; $100a^4b^4y^2$; $16\,(a-b)^4$.

3. $\dfrac{9a^2x^4}{16y^4}$; $\dfrac{4a^6b^2}{9x^4z^4}$; $\dfrac{a^4x^4y^2}{m^4x^2}$; $\dfrac{25a^6b^4}{16m^4n^2}$; $\dfrac{169a^4b^8}{16x^2y^8}$.

Find the roots of the following, and simplify :—

4. $3\sqrt{16}+5\sqrt{36}-3\sqrt[3]{27}+\sqrt{225}-4\sqrt[3]{343}+2\sqrt[3]{512}.$

5. $\sqrt{\dfrac{a^4b^4}{16}}$; $\sqrt[3]{\dfrac{8a^3}{b^3}}$; $\sqrt[3]{\dfrac{-8a^3}{b^3}}$; $\sqrt[3]{\dfrac{64b^6c^3}{125a^6}}$; $\sqrt[3]{-\dfrac{216a^3b^3c^6}{343}}$.

6. $\sqrt[4]{a^4b^8}$; $\sqrt[4]{16x^8y^4}$; $\sqrt[4]{\dfrac{81a^4b^8c^{12}}{256m^8}}$; $\sqrt[4]{\dfrac{256a^4b^{12}c^{16}}{625x^4y^{12}}}$.

7. $\sqrt[5]{a^5b^{10}}$; $\sqrt[5]{-\dfrac{32}{a^{15}}}$; $\sqrt[5]{\dfrac{243x^5y^{10}}{32z^{10}}}$; $\sqrt[5]{-\dfrac{1024a^{15}b^{10}}{32x^{10}y^5}}$; $\sqrt[6]{\dfrac{64x^{12}y^6}{a^6b^{12}}}$.

(*b*)

Find the square roots of :—

1. $a^2+2ab+b^2$; $1-2x+x^2$; $1+2x^2+x^4$.

2. $a^4+4a^2b^2+4b^4$; $4a^4-4a^2+1$; $x^2-12ab+36b^2$.

3. $1-6x^2+9x^4$; $4-12x+9x^2$; $\dfrac{1}{4}-x^2+x^4$.

4. $9a^4-24a^2b^3+16b^6$; $x^2+5x+\dfrac{25}{4}$; $a^4-a^2+\dfrac{1}{4}$.

5. $x^2+2+\dfrac{1}{x^2}$; $x^4+2+\dfrac{1}{x^4}$; $\frac{9}{4}a^{12}-2a^6n^4+\frac{4}{9}n^8$

6. $a^2+2ab+b^2+2bc+c^2+2ac$; $(a+b)^2+2c\,(a+b)+c^2$.

7. $a^2+2ab-2ac+b^2-2bc+c^2$; $x^2+4y^2+9z^2-4xy+12yz-6xz$.

8. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$; $a^4 + 2a^3b + 3a^2b^2 + 2ab^3 + b^4$.

9. $1 + 4x + 10x^2 + 12x^3 + 9x^4$; $9a^4 + 12a^3 + 22a^2 + 12a + 9$.

Find the cube roots of :—

10. $a^3 + b^3 + 3a^2b + 3ab^2$; $1 - 3x + 3x^2 - x^3$.

11. $8 + 12x + 6x^2 + x^3$; $27 - 27x + 9x^2 - x^3$.

12. $x^3 + 12x^2 + 48x + 64$; $8x^3 - 36a^2b + 54ab^2 - 27b^3$.

13. $a^3 + 24a^2b + 192ab^2 + 512b^3$; $8a^3 - 84a^2b + 294ab^2 - 343b^3$.

14. $(3a - a^3)^2 + (1 - 3a^2)^2$.

15. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

(*c*)

Ascertain the square roots of :—

1. $\frac{9a^2}{25} - 2 + \frac{25}{9a^2}$; $\frac{a^2}{4b^2} - \frac{2a}{b} + 4$; $\frac{64m^2}{9w^2} + 4 + \frac{32m}{3w}$.

2. $\frac{x^4}{64} + \frac{x^3}{8} - x + 1$; $x^4 + 2x^3 - x + \frac{1}{4}$; $x^4 + 4x^3 + 6x^2 + 4x + 1$.

3. $4x^2 + 9y^2 + 25z^2 + 12xy - 30yz - 20zx$; $4x^4 - 4x^3 + 5x^2 - 2x + 1$.

4. $24 + \frac{16y^2}{x^2} - \frac{8x}{y} + \frac{x^2}{y^2} - \frac{32y}{x}$; $x^2 + 2 + \frac{1}{x^2} - 4x + \frac{4}{x}$.

5. $\frac{9a^2}{b^2} - \frac{6a}{5b} + \frac{101}{25} - \frac{4b}{15a} + \frac{4b^2}{9a^2}$. N. B. $\frac{101}{25} = 4 + \frac{1}{25}$.

Find the cube roots of :—

6. $64a^3 - 144a^2b + 108ab^2 - 27b^3$; $54 - 27x^3 + \frac{8}{x^6} - \frac{36}{x^3}$.

7. $1 - 6x + 21x^2 - 44x^3 + 63x^4 - 54x^5 + 27x^6$; $\frac{x^3}{8} - \frac{3x^2}{4} + \frac{3x}{2} - 1$.

8. $a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3$.

9. $\frac{x^3}{27} - \frac{x^2}{3} + 2x - 7 + \frac{18}{x} - \frac{27}{x^2} + \frac{27}{x^3}$; $\frac{x^3}{27} + \frac{2x^2}{3} + 4x + 8$.

10. If $a + \frac{1}{a} = x$, show that $a^3 + \frac{1}{a^3} = x^3 - 3x$. See Art. 91 (G) (2).

CHAPTER VIII.

SYMMETRY.

93. An expression is said to be **Symmetrical** with respect to *two* of its letters when these can be interchanged by substituting each for the other without altering the value of the expression.

Thus $a + b$, $a^2 + ab + b^2$, $a^3 + a^2b + ab^2 + b^3$ are symmetrical with respect to a and b, for on substituting a for b and b for a, these expressions become $b + a$, $b^2 + ba + a^2$, $b^3 + b^2a + ba^2 + a^3$, which differ from the given expressions only in the order of their terms and of their factors. So also $x^2 + y^2x + yb + b^2x$ is symmetrical with respect to y and b, for on substituting y for b and b for y, it becomes $x^2 + b^2x + by + y^2x$, which is identical with the given expression.

On interchanging x and y, $x^2 + y^2x + yb + b^2x$ becomes $y^2 + x^2y + xb + b^2y$; this is *not* the same as the original expression, which is, therefore, *not* symmetrical with respect to x and y. In the same way it may be shown that this expression is not symmetrical with respect to x and b.

94. An expression is symmetrical with respect to *three* of its letters a, b, c, if it remains the same when a is changed into b, b into c, and c into a. Thus,

$a^2b + b^2c + c^2a$ is symmetrical with respect to the set of letters (called *Cycle*) abc, for on changing a into b, b into c, and c into a, it becomes $b^2c + c^2a + a^2c$, which differs from the original expression only in the order of its terms. So also $(a - b)^3 + (b - c)^3 + (c - a)^3$ is symmetrical with respect to a, b, c, for on interchanging the letters we get $(b - c)^3 + (c - a)^3 + (a - b)^3$ which differs from the original expression only in the order of its terms. So also $(x - a)(a - b)^2 + (x - b)(b - c)^2 + (x - c)(c - a)^2$ is symmetrical with respect to a, b, c.

Generally, an expression is symmetrical with respect to any set of letters, a, b, c, ... , h, k, called the *cycle* $(abc \ldots hk)$, if it remains the same expression when a is changed into b, b into c, ..., h into k, and k into a.

Thus, $ab + bc + cd + da$ and $ac + bd$ are each symmetrical with respect to the cycle $(a\,b\,c\,d)$.

95. Principle of Symmetry.—*An expression which in any one form is symmetric with respect to any set of letters will, in every other form, be symmetric with respect to the same set of letters.* Thus $a^3 + b^3 + c^3 - 3abc$ is symmetric with respect to a, b and c, it will therefore remain symmetric when written in any other form, as for instance, in the form

$$\tfrac{1}{2}(a + b + c)\{(a - b)^2 + (b - c)^2 + (c - a)^2\}.$$

96. The kind of symmetry defined in the preceding articles is strictly speaking *Cyclo-symmetry; i. e.*, it is true only when *all the letters of the set* (*cycle*) are interchanged. An expression is **completely symmetric** with respect to three or more of its letters when it is symmetric with respect to *each and every pair* of these that can be selected. Thus $x^3 + y^3 + z^3 - 3xyz$ is *completely symmetric* with respect to x, y, z, for it remains the same on interchanging x and y, or y and z, or z and x; and these are all the pairs that can be selected from x, y, and z. So also $a^2b + b^2a + a^2c + c^2a + b^2c + c^2b$, is *completely symmetric* with respect to a, b, and c, for any *pair* of them may be interchanged without altering the expression. So also are $ab + ac + ad + bc + bd + cd$, and $bcd + acd + abd + abc$, which $= abcd\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)$, for in each case any *pair* may be interchanged without changing the value of the expression. But $a^2b + b^2c + c^2a$ and $ab + bc + cd + da$, are *not* completely symmetric, for the interchange of any pair of letters alters the value of the expressions.

It is clear, then, that an expression may be symmetric as defined in the preceding article (*i. e.*, cyclo-symmetric) with respect to a set of letters, and at the same time *not* be *completely* symmetric. But *every expression which is completely symmetric* with reference to a set of letters *is necessarily cyclo-symmetric* with respect to the same letters. Thus $a^2b + b^2c + c^2a$ is, as we have seen, cyclo-symmetric, but not completely symmetric; but $x^3+y^3+z^3-3xyz$, which is *completely* symmetric, is also cyclo-symmetric. We use symmetry as = cyclo-symmetry in the following pages.

97. In an expression which is symmetric with respect to certain letters, any term of a *group of terms* which are formed according to the same law, may be called a *type of that group*, or simply a *type-term*. Thus with respect to the set a, b, c, ab may be called the *type* or *type-term of the products of every pair of the letters*, *i. e.*, of ab, bc, ca. So also a^2b is the type of the group a^2b, b^2c, c^2a. Again, with respect to four letters, a, b, c, d, ab is the *type-term* of ab, ac, ad, bc, bd, cd; and abc, the type-term of the products formed by every three of the letters, that is of abc, abd, bcd, cda. Such symmetric *groups* are, for the sake of brevity, frequently indicated by prefixing the Greek letter Σ (sigma) to a type-term of the group; thus instead of $a+b+c+\ldots.k$, we write Σa. With respect to the four letters a, b, c, d, Σabc denotes $abc + acd + bcd + cda$. The symbol (Σ) may be read "*the sum of all such terms as.*"

Thus Σa = the sum of all such terms as a.
Σa^2b = " " " " " " " a^2b.
Etc., etc.

EXERCISE XXXVII.

Write the following in full:—

1. Σa^2b; $\Sigma a(a+b)^2$; $\Sigma ab(b-c)$; Σa^2bc; $\Sigma a(b+c)$.

2. $\Sigma(a-b)(b-c)$; $\Sigma a^2(b-c)$; $\Sigma a(b-c)^2$; $\Sigma(x-a)(b-c)^2$.

3. Σa^3; Σa^2bc; Σa^2b; $\Sigma(a+b)$; Σab; $\Sigma a^2(a-b)$; $\Sigma(a-b)^3$,—each with respect to the letters a, b, c, d.

4. $\Sigma(a-b)^2(b-c)^2$; $\Sigma(x-a)(b-c)^2$; and $\Sigma b^2(c-a)$—each with respect to the letters a, b, c, and d.

Show that the following are symmetrical:—

5. $x^2 + x^2y + xy^2 + y^2$, with respect to x and y.

6. $(x+y)^2 + (x-y)^2$, with respect to x and y, and also with respect to x and $-y$.

7. $a^3 + b^3 + c^3 - 3abc$, and $(a-b)^2 + (b-c)^2 + (c-a)^2$, with respect to a, b, and c.

8. $(x+y+z)^3 - (x^3+y^3+z^3)$, and $(x-y)^3 + (y-z)^3 + (z-x)^3$, each with respect to x, y, and z.

9. $a(b-c)^3 + b(c-a)^3 + c(a-b)^3$, with respect to a, b, and c.

10. $(ac+bd)^2 + (bc-ad)^2$, with respect to a^2 and b^2, and also with respect to c^2 and d^2.

11. $\frac{1}{2}(a+b+c)\{(a-b)^3 + (b-c)^2 + (c-a)^2\}$, with respect to a, b, and c.

12. $x(x+2y)^3 + y(y+2x)^3$, with respect to x and y, and also x and $-y$.

With respect to what letters are the following expressions symmetrical?—

13. $a^3 + b^3 - c^3 + 3abc$; $a^3 - b^3 + c^3 + 3abc$; $a^3 - b^3 - c^3 - 3abc$.
a, b, -c, a, -b, c a, -b, -c

14. $a^2b^2 + b^2c^2 + c^2a^2 + 2abc(a+b+c)$; $abc + 5ab + 2(a^3+b^3)$.

15. $bc(b-c) - ca(a-c) - ab(b-a)$; $(a-b)^4$; $(a-b)^3$.

a, b a, -b

16. $(a-b)^5 + (b-c)^5 + (c-a)^5$; $2(a^2x^2 + b^2y^2) - 2ab(ax+by+xy)$.

ax by

17. $\dfrac{1}{x+a+b} + \dfrac{1}{x-a+b} + \dfrac{1}{x+a-b} - \dfrac{1}{a+b-x}$; a & b
$(a-x)(b-x)(x-c) + abc$. a, b, c

18. $(a+b+c)^3 - 3\{a(b-c)^2 + b(c-a)^2 + c(a-b)^2\}$. a, b, c

19. $\dfrac{1}{x-a} + \dfrac{1}{x-b} = \dfrac{a+b}{ab}$; $\dfrac{bx}{x+b} + \dfrac{ay}{x+a} = \frac{1}{2}(a+b)$.
a, - a, b

20. $(b-c)(x-b)(x-c) - (a-c)(x-c)(x-a) - (a-b)(a-x)(b-x)$.
a, b, c -(b-a)(a-x)(b-x

Give the *type-terms* in the following expressions :—

21. $bc^2 + cb^2 + bc(b+c) - bc(b+c) - bc(b-c)$.

22. $a^4 + a^3b + a^2b^2 - ab^3 + abc^2 - ab^2c$.

23. $x^3 + y^3 + z^3 + 3(x+y)(y+z)(z+x)$.

24. $a^3(b-c) + b^3(c-a) + c^3(a-b)$.

25. $a(b-c)^3$ + etc. + etc.

26. $x(y+z)^2 + y(z+x)^2 + z(x+y)^2 - 12xyz$.

27. $(x+y)^3$, $(x+y)^4$, $(x-y)^5$.

28. $(x+y+z)^3 + (x+y-z)^2 + (y+z-x)^3 + (z+x-y)^3$.

29. $(a+b+c)^4-(a^4+b^4+c^4)$; $(x+y+z)^5-(x^5+y^5+z^5)$.

30. $(a+b+c+d)^3$; $(x+y)^7-(x^7+y^7)$.

31. $a^3+b^3+c^3-3abc$ is symmetrical in a, b, and c; write down an expressson *formed by the same law* from a, b, c, and d.

32. If $a+b+c$ exactly divides $a^3+b^3+c^3-3abc$, what are the exact divisors of $a^3+b^3-c^3+3abc$, of $a^3-b^3+c^3+3abc$, of $-a^3+b^2+c^3+3abc$, and of $a^3-b^3-c^3-3abc$?

33. In the last question if the quotient in the *first* case is $\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}$, what is the quotient in each of the other cases?

98. By the principle of symmetry, an expression which is symmetric in any form with respect to any letters, will be symmetric in every other form with respect to the same letters. This principle may be used with advantage in changing such an expression from *one form* to another; we have to determine the *type-terms* and then, from these, the other terms can be written down at once.

Ex. 1. Simplify $(a+b-2c)^2+(b+c-2a)^2+(c+a-2b)^2$.

The only type-terms are a^2 and ab; of these we have

a^2 in 1st bracket $+\ 4a^2$ in 2d bracket $+\ a^2$ in 3d bracket $=6a^2$, and
$2ab$ " " $-\ 4ab$ " " $-\ 4ab$ " " $=-6ab$.

$\therefore$ the result is $6(a^2+b^2+c^2-ab-bc-ca)$, or $6\Sigma a^2-6\Sigma ab$.

Ex. 2. Expand $(x+y)^3$.

Since the expression is of *three* dimensions the *only* type-terms are x^3 and x^2y; *i. e.*, the literal parts are x^3, y^3, x^2y, xy^2. Hence,

$$(x+y)^3=x^3+y^3+n(x^2y+xy^2), \text{ where } n \text{ is } \textit{numerical}.$$

This is an *identity*, and is therefore true for all values of x and y. Let $x=y=1$ in this identity and we have

$$(1+1)^3=1+1+n(1+1); \quad \therefore n=3, \text{ and}$$

$$(x+y)^3=x^3+y^3+3(x^2y+y^2x)=x^3+y^3+3xy(x+y).$$

Ex. 3 . Expand $(x+y+z)^3$. The only type-terms are x^3, x^2y, xyz. By Ex. 2° we know that the expression contains x^3, $3x^2y$, and $\therefore 3x^2z$; *i. e.*, $x^3 + 3x^2(y+z)$, hence,

$$\begin{aligned}(x+y+z)^3 = x^3 &+ 3xy\,(y+z)\\ +\,y^3 &+ 3yz\,(z+x)\\ +\,z^3 &+ 3zx\,(x+y)\\ &+ \quad n\,(xyz),\end{aligned}$$ in which n is numerical.

By putting $x=y=z=1$ we get

$$(1+1+1)^3 = 1+1+1+3(1+1)+3(1+1)+3(1+1)+n(1);\ \therefore n=6.$$

Ex. 4 . Simplify $(a+b+c)^2+(a+b-c)^2+(b+c-a)^2+(c+a-b)^2$.

The *type-terms* are a^2 and ab. We get from the several brackets

$$a^2+a^2+a^2+a^2=4a^2,$$

and $$+2ab+2ab-2ab-2ab=0;$$

$$\therefore \text{Expression} = 4(a^2+b^2+c^2).$$

Ex. 5 . $a+b+c$ is found to be a factor of $a^3+b^3+c^3-3abc$. What is the other factor?

We obtain by division the first *two* terms of the required factor; they are a^2 and $-ab$; $\therefore$ the factor required is

$$a^2+b^2+c^2-ab-bc-ca.$$

Ex. 5 . Expand $(a+b+c+d)^3$. From Ex. 3°, we get

$$\begin{aligned}&a^3+3a^2(b+c+d)+6bcd, \text{ and}\\ &b^3+3b^2(c+d+a)+6cda\\ &c^3+3c^2(d+a+b)+6dab\\ &d^3+3d^2(a+b+c)+6abc.\end{aligned}$$

Ex. 6 . Obtain the simplest form of

$$(a+b+c)(x+y+z)+(a+b-c)(x+y-z)+(a-b+c)(x-y+z)\\ +(-a+b+c)(-x+y+z).$$

We have to calculate only the type-terms ax, ay, az. We get

ax from each of the four products; *i. e.*, $4ax$.

$ay+ay-ay-ay$ from the four products; *i. e.*, $2ay-2ay=0$.

$az-az+az-az$ " " " " *i. e.*, $2az-2az=0$.

Hence the given expression $= 4ax+4by+4cz$.

EXERCISE XXXVIII.

Simplify the following :—

1. $(a-b-c)^2+(b-c-a)^2+(c-a-b)^2$.
2. $a(b-c)+b(c-a)+c(a-b)$.
3. $(x+y+z)^2-x(x+y-z)-y(y+z-x)-z(z+x-y)$.
4. $(a-b-2c)^2+(b-c-2a)^2+(c-a-2b)^2$.
5. $(x-y)^2+(y-z)^2+(z-x)^2$.
6. $(a+2b-3c)^2+(b+2c-3a)^2+(c+2a-3b)^2$.
7. $(a+b+c+d)^2+(a-b-c+d)^2+(a-b+c-d)^2+(a+b-c-d)^2$.
8. $(ax+by+cz)^2+(ax+by-cz)^2+(by+cz-ax)^2+(cz+ax-by)^2$.
9. $(a+b+c)^3+(a+b-c)^3+(b+c-a)^3+(c+a-b)^3$.
10. $a(a+b+c-d)+b(b+c+d-a)+c(c+d+a-b)+d(d+a+b-c)$.
11. $(a+b)^2+(a+c)^2+(a+d)^2+(b+c)^2+(b+d)^2+(c+d)^2$.
12. $(a^2-bc)(b-c)+(b^2-ca)(c-a)+(c^2-ab)(a-b)$.
13. $a(b+c)^2+b(c+a)^2+c(a+b)^2-c^2(a+b)-b^2(c+a)-a^2(b+c)$.
14. $(ab+bc+ca)^2-(a^2b^2+b^2c^2+c^2a^2)-(a^2bc+b^2ca+c^2ab)$.
15. $(x+y+z)^4+(x+y-z)^4+(y+z-x)^4+(z+x-y)^4$.
16. Show that $(a^2+2bc)^3+(b^2+2ca)^3+(c^2+2ab)^3-3(a^2+2bc)$ $(b^2+2ca)(c^2+2ab)=(a^3+b^3+c^3-3abc)^2$.
17. If $2s=a+b+c$, show that $(s-a)(s-b)+(s-b)(s-c)$ $+(s-c)(s-a)=s^2-\frac{1}{2}(a^2+b^2+c^2)$.
18. If $p=b+c+d-a$, $q=c+d+a-b$, $r=d+a+b-c$, $s=a+b+c-d$, find Σrs, in terms of a, b, c, d.
19. If $s=a+b+c+d$, then $\dfrac{s-a}{a}+\dfrac{s-b}{b}+\dfrac{s-c}{c}+\dfrac{s-d}{d}$ $=s\cdot\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}\right)-4$.
20. Prove $(a-b)^4+(b-c)^4+(c-a)^4=\frac{1}{2}\{(a-b)^2+(b-c)^2+(c-a)^2\}^2$.
21. Show that $(av+bx+cy+dz)^2+(ax-bv+cz-dy)^2+(ay-cv-bz$ $+dx)^2+(az-dv+by-cx)^2=(a^2+b^2+c^2+d^2)(v^2+x^2+y^2+z^2$.
22. Prove that $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$, where $2s=a+b+c$.

CHAPTER IX.

RESOLUTION INTO FACTORS.

99. In Multiplication we have to determine the *product* when the factors are given; the converse process of determining the factors when the product is given, is called **Resolution into Factors.** Some of the methods of doing this will now be illustrated.

100. Monomial Factors.—When *every term* of an expression contains the same monomial factor, this may be separated by *division*. (See Art. 64.)

Ex. 1. $a^2 - ab = a(a - b)$; $a^2b + ab^2 + a = a(ab + b^2 + 1)$.

Ex. 2. $a^2x^2 + a^2b^2x^3 - a^2bx^4$. Here a^2x^2 is contained in *every* term, we therefore *divide* by this factor and write the *quotient* in brackets as the other factor; thus,

$$a^2x^2(1 + b^2x - bx^2).$$

Ex. 3. $ab^2c - a^2bc + c^2 = c(ab^2 - a^2b + c)$.

Ex. 4. $6x^2y - 9x^2y^2 + 12xy^3 = 3xy(2x - 3xy + 4y^2)$.

Ex. 5. $a^3x^4 - a^nx^n = a^3x^4(1 - a^{n-3}x^{n-4})$, if $n > 4$.

Ex. 6. $6(a + b)x^2y - 18(a + b)xy^2 = 6xy(a + b)(x - 3y)$.

EXERCISE XXXIX.

Resolve into factors:—

1. $a^2 + a$; $a^2 + ab$; $a - a^2$; $a^2 - a^3$; $a^2b + ab^2$; $x^3y - xy^4$; $a - a^2b^2$; $a^2b^2 + ab$; $m^2p - m^2p^3$.

2. $ab - ac$; $bc - bc^2$; $b^2 - ba$; $c^2 - ca$; $ab^2 - abc$; $a + a^2 + a^3$; $a^2 - a^3 + a^4$; $a^2b - ab^2 + a^2$; $x + x^2 - x^3$.

3. $ax^3 - bx^2$; $a^3b^3 - ab^3$; $xy - xy^2 + x^2y$; $ab - a^2b + a^2b^2$; $xy - y^2 + y$; $a^2b^2 + b^2c^2 - a^2b^2c^2$.

4. $a - a^2xy$; $b^2xy - c^2xy + a^2x^2y^2$; $a^3b^2cy + a^2b^3y + ab^3c^3y$.

5. $6ax^2 + 9a^2x - 3a^2x^2$; $a^3b^2x + a^3b^2x^3$; $7 - 14ab + 21abc$.
6. $3x^3a - 3x^2a^3b - 6a^3x^3$; $16x^2y^3 - 8x^3y^2$; $a^4b - b^4a$.
7. $5x^5 - 10x^3a^3 + 15a^3x^3$; $3 - 3a^2x^2 + 9a^2x^3$; $a^nb - b^na$.
8. $(a-b)m - (a-b)n$; $5(x+y)c^2 - 10(x+y)a^2$; $a^{2n}b^n + a^nb^{2n}$.
9. $(a+b)^2 - (a+b)^2x$; $(x-y)^2a^2b^2 - b^2c^2(x-y)^2$.
10. $p^2(a-b)^n - q^2(a-b)^n + r^2(a-b)^n$; $27(a-b)^6 - 9(a-b)^3 + 3(a-b)^2$.
11. $a^nb^{2n}c^{3n} + a^{2n}b^{3n}c^n - a^{3n}b^nc^{3n}$;
$x^ny^n(2a-b) + x^ny^n(2b-c) - x^ny^n(2c-a)$.

101. By the same principle an **expression of four or more terms** may sometimes be resolved by taking it in parts. (Art. 43.) Thus,

Ex. 1. Just as
$$6x + 6y + 4x + 4y = (6+4)x + (6+4)y = (6+4)(x+y).$$
So
$$ax + ay + bx + by = (a+b)x + (a+b)y = (a+b)(x+y).$$

Ex. 2. $ac + by + ay + bc = c(a+b) + y(a+b) = (a+b)(c+y)$.

Ex. 3. $a^2 - ac + ab - bc = a(a-c) + b(a-c) = (a+b)(a-c)$.

Ex. 4.
$$acx^2 + adx - bcx - bd = ax(cx+d) - b(cx+d) = (ax-b)(cx+d).$$

EXERCISE XL.

(a)

Resolve into factors :—

1. $ax + 2ay - bx - 2by$.
2. $2ax - 3by + 2bx - 3ay$.
3. $a^2 - ab + ax - bx$.
4. $abc - c^2 - abd + cd$.
5. $mx^2 - ma + nx^2 - na$.
6. $a^2 - ac + ab - bc$.
7. $3ax + ay + 3bx + by$.
   $x+y)(a+b$
8. $a - bc - ax + bcx$.
9. $ac - by + ay - bc$.
   $(a-$
10. $a^2 + ab + ax + bx$.
11. $3ax - bx - 3ay + by$.
    $a-b \quad x-$
12. $7a - 7bc - ax + bcx$.
    $(7-x \quad a-b$
13. $3pr + qr - 3ps - qs$.
    $(3p+q)(r-s)$
14. $1 - a - b + ab$.
15. $6x^2 + 3xy - 2ax - ay$.
16. $a^3 + a^2 - a - 1$.
    $(a^2-1$
17. $3bx^2 - 3bx + 1 - x$.
    $(3bx-1 \quad x-$
18. $axy + bcxy - az - bcz$.
    $(a+b$

(*b*)

19. $a^3 - 1 - a^2 + a$.

$(a^2+1)(a-1$

20. $2af + bx + 2ax + bf$.

21. $ax^2 - 3cx^2 + a^3 - 3a^2c$.

$(a-3c)(x^2+a^2)$

22. $x^2 + 3y - 3x - xy$.

23. $2a^2x^2 + 1 - 2a^2 - x^2$.

24. $bc + 1 - b - c$.

25. $a^4x^4 - a^2bx^2 + bc - a^2cx^2$.

$(a^2x^2 - b$

26. $3a^2 + 3b^2 - 3a^2b^2 - 1$.

$3b^2 - (1-3a^2)$

27. $(x^2 - a^2)(x^2 + a^2) + ax^3 - a^3x$.

$x^2 - a$

28. $ax + by + az - bx - ay - bz$.

$(a-b)(x-y+z)$

29. $a^2x + abx + ac + aby + b^2y + bc$.

$a+b$

30. $abx^{2n} - b^2x^n - ab + a^2x^n$.

$(ax^n - b)$

31. $(a + 1)(a - 1) + ab + 1 - b - a$.

32. $6 - 2y^n - x^ny^n + 3x^n$.

33. $a - b + c - ab - bc + b^2$.

34. $2apq - 3abf - 2pqx + 3bfx$.

$(a-x)(2pq-3bf)$

35. $(1 + x^2)(1 - x^2) + q - px^2 - qx^2 + p$.

36. $2p^nr^n - 4p^ns^n + 6s^nq^n - 3q^nr^n$.

37. $ad + db + ce - ae + bf - cf + af - cd - be$.

$a+b-c \quad (d-e+f$

38. $1 - a + b + p + q - ap + bp - aq + bq$.

$1-a+b \quad (1+$

102. The expression for the square of a binomial (Chap. IV. form A) helps us to find the factors of a **trinomial which is a perfect square.** We have merely to *connect the square roots of the* **two squares** *in the trinomial by the sign of the other term*, and the result will be one of the equal factors. In a similar way we may find the equal factors of **any polynomial** which is a perfect square (Art. 85).

Ex. 1 . $a^2 - 4ab + 4b^2 = (a - 2b)^2 = (a - 2b)(a - 2b)$.

Ex. 2 . $36a^2b^2 - 24abc + 4c^2 = (6ab - 2c)(6ab - 2c)$.

Here the square root of the first square is $6ab$, of the second $2c$, and the sign of the *other* term $(-24abc)$ is *minus*. ∴ Connect $6ab$ and $2c$ by the sign $-$, and we have one of the equal factors, $(6ab - 2c)$.

Ex. 3 . $4 - 16m^2n^2 + 16m^4n^4 = (2 - 4m^2n^2)(2 - 4m^2n^2)$.

Ex. 4 . $(x + y + z)^2 - 6(a + b)(x + y + z) + 9(a + b)^2$
$= \{(x + y + z) - 3(a + b)\}\{(x + y + z) - 3(a + b)\}$.

Ex. 5 . $4a^4 + 4b^4 + 4c^4 - 8a^2b^2 - 8a^2c^2 + 8b^2c^2$.

In this example we see *three squares*, and three *double* products; it is, therefore, *probably* the square of a trinomial. Take the square root of each *square*, and we have, $2a^2$, $2b^2$, $2c^2$; now by what signs are these to be connected? We determine these signs from an *inspection of the signs of the double products;* we have

$-8a^2b^2$ ∴ the signs of a^2 and b^2 will be *different*;
also, $-8a^2c^2$ ∴ " " " a^2 and c^2 " " "
and, $+8b^2c^2$ ∴ " " " b^2 and c^2 " " *alike.*

Hence we have :—

$$(2a^2 - 2b^2 - 2c^2)\,(2a^2 - 2b^2 - 2c^2).$$

Ex. 5. $4a^2 - 12ab + 24b - 16a + 9b^2 + 16$.

The *squares* are $4a^2$, $9b^2$ and 16; ∴ the *terms* are $2a$, $3b$, and 4, and the expression

$$= (2a - 3b - 4)\,(2a - 3b - 4).$$

EXERCISE XLI.

Resolve into factors :—

1. $a^2 - 2ab + b^2$; $1 - 2x + x^2$; $4 - 8y + 4y^2$; $9 - 6x + x^2$; $x^2 - 2xy + y^2$; $1 - 2y + y^2$.
2. $x^2 - 4xy + 4y^2$; $1 - 6x + 9x^2$; $a^4 + 2a^2b^2 + b^4$; $1 - 2x^2 + x^4$; $4y^4 - 4y^2 + 1$.
3. $4a^2 - 12ab + 9b^2$; $a^2 + 12ab + 36b^2$; $1 - 2x^3 + x^6$.
4. $9a^2 + 12a + 4$; $16x^2y^2 + 24xyz^2 + 9z^4$; $1 - 5y + 6\frac{1}{4}y^2$.
5. $a^2 + 2abc + b^2c^2$; $9x^2 - 6xy + y^2$; $16a^4 - 24a^2x^3 + 9x^6$.
6. $1 + 4y^2 + 4y^4$; $25 - 40x + 16x^2$; $a^4b^2 + 4a^2b^3c^2 + 4b^4c^4$.
   $(5 - 4x)$ $(a^2b + 2b^2c^2)$
7. $49x^6 - 42x^3yz^2 + 9y^2z^4$; $9a^2x^2 - 6ax + 1$; $\frac{1}{4} - y + y^2$.
   $3a$ $- 1$
8. $x^4y^4 - x^2y^2 + \frac{1}{4}$; $\frac{1}{4}a^4 + 16b^2c^2 - 4a^2bc$; $\frac{1}{4}a^4 - \frac{1}{3}a^2b^2c^2 + \frac{1}{9}b^4c^4$.
9. $(a - b)^2 + 2\,(a^2 - b^2) + (a + b)^2$: $x^2 - 2x\,(z - y) + (z - y)^2$.
10. $(p + q + r)^2 - 2s\,(p + q + r) + s^2$; $a^2x^2 - 2axy^2 + y^4$: $a^2 - 2ab^n + b^{2n}$.
11. $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$; $1 - 2x^m + x^{2m}$; $(2x - 3y)^2 + (2x + 3y)^2 - 2\,(4x^2 - 9y^2)$.
   $\left(\frac{a}{b} - b\right)^2$ $1 - x$

12. $(a^2 + ab + b^2)^2 - 2(a^4 + a^2b^2 + b^4) + (a^2 - ab + b^2)^2$; $(2a\ [illegible])^2$
$\left(\frac{x}{y}\right)^{2m} + \left(\frac{y}{x}\right)^{2m} - 2$. $\left\{\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right\}^2$

13. $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$; $p^2 + q^2 + r^2 - 2pq - 2qr + 2pr$.
$(x + y + z)^2$ $(p - q + r)^2$

14. $a^2 + 4b^2 + 9c^2 - 4ab - 12bc + 6ac$; $1 + x^2 + y^2 - 2x - 2xy + 2y$.
$(a - 2b + 3c)^2$ $(1 - x + y)^2$

15. $9a^2 + 12ab + 4b^2 + 6ac + 4bc + c^2$; $4a^4 - 12a^3 + 25a^2 - 24a + 16$. (Observe, $25a^2 = 9a^2 + 16a^2$).
$(3a + 2b + c)^2$ $(2a^2 - 3a + 4)^2$

16. $9a^2x^2 + 12abxy + 4b^2y^2 + 6acxz + c^2z^2 + 4bcyz$.
$(3ax + 2by + cz)^2$

17. $4a^4 - 12a^2b + 9b^2 + 16a^2c + 16c^2 - 24bc$; $a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 + 2b^2c^2$.
$(2a^2 - 3b + 4c)^2$ $(-a^2 + b^2 + c^2)^2$

Find the quantity which must be added to each of the following expressions to make it a perfect square; also write the factors of the completed squares:—

18. $x^2 + 4y^2$; $x^2 + \frac{1}{4}y^2$; $x^4 + x^2y^2 + y^4$; $4x^2 - 10xy + 25y^2$; $x^4 + 4y^4$.
$\pm xy$ $-10xy$

19. $4a^2 + 9y^2$; $a^4 + 25a^2b^2$; $9x^6 + 4y^4$; $a^{2n} + b^{2n}$; $x^2 + 2ax$.
± 6 ± 12 $\pm 2a^nb^n$ a^2

20. $x^4 - 2a^2x^2$; $4y^2 - 4yz$; $y^2 - y$; $x^2 + x$; $x^2 + 4x$; $x^2 + 2bx$.
a^4 $\frac{1}{4}$ 4 b^2

21. $x^2 + bx$; $y^4 + 4b^2x^2$; $\left(\frac{a}{b}\right)^2 + \left(\frac{b}{a}\right)^2$; $\left(\frac{a}{b}\right)^{2m} + \left(\frac{b}{a}\right)^{2m}$; $\frac{1}{4} - x^2$.
$\pm$ $\pm$ $\pm$ x

22. $x^2 - 2x - 6$; $4x^2 + 2x$; $a^2x^2 + bx$; $x^2 - 5x$; $x^2 - 7x$.
7 $b^2 \div 4a$ $25 \div$ 4

23. $4x^2 - 9x$; $3x^2 + 8x$; $3x^2 + 8x - 9$; $x^2 + bx + c$.
$x^2 + 4$ $x^2 + 13$ $-c +$

103. Since $(x + a)(x - a) = x^2 - a^2$ (formula B) it follows that **the difference of any two squares** can always be resolved into factors. We have merely to *take the square root of each of the squares, and connect these roots by the sign + for one of the factors, and by the sign − for the other.*

Ex. 1. $x^2 - 4a^2 = (x + 2a)(x - 2a)$; $4x^2 - 9a^2 = (2x + 3a)(2x - 3a)$; $1 - 4x^2 = (1 + 2x)(1 - 2x)$.

Ex. 2. $16a^2b^2 - 4x^2y^2 = (4ab + 2xy)(4ab - 2xy)$.

Here the first term has the two equal factors each $4ab$; and the second has the two equal factors each $2xy$; we connect these by the sign + for one factor, and by the sign − for the other.

Ex. 3 . $(2a+b)^2 - c^2 = \{(2a+b)+c\}\{(2a+b)-c\}$
$= (2a+b+c)(2a+b-c)$.

Ex. 4 . $(1-x+x^2)^2 - (1+x+3x^2)^2 = (2+4x^2)(-2x-2x^2)$.

Here the roots are $(1-x+x^2)$ and $(1+x+3x^2)$.

Connecting these by the sign +, and reducing, we have $2+4x^2$; and connecting them by the sign −, and reducing, we have $-2x-2x^2$.

Ex. 5 . $(x^2+y^2)^2 - (x^2-y^2)^2 - (x^2+y^2-z^2)^2$: Take the first two together, thus :—

$(x^2+y^2)^2 - (x^2-y^2)^2 = (x^2+y^2+x^2-y^2)(x^2+y^2-x^2+y^2)$
$= (2x^2)(2y^2) = 4x^2y^2$; now take this result with the third square in the given expression, thus :—

$$4x^2y^2 - (x^2+y^2-z^2)^2 = \{2xy+(x^2+y^2-z^2)\}\{2xy-(x^2+y^2-z^2)\}$$
$$= \{2xy+x^2+y^2-z^2\}\{2xy-x^2-y^2+z^2\}.$$

The first of these factors $= (x+y)^2 - z^2 = (x+y+z)(x+y-z)$
" second " " $= z^2-(x-y)^2 = (z+x-y)(z-x+y)$;

hence the given expression

$$= (x+y+z)(x+y-z)(z+x-y)(z-x+y).$$

Ex. 6 . $405a^4 - 80b^4 = 5(81a^4 - 16b^4)$
$= 5(9a^2+4b^2)(9a^2-4b^2) = 5(9a^2+4b^2)(3a+2b)(3a-2b)$,

since $9a^2 - 4b^2 = (3a+2b)(3a-2b)$.

This is an example showing that sometimes we have to **separate a monomial factor**, in order to apply the formula.

Ex. 7 . $16x^7 - 81x^3 - 16x^4 + 81$: This expression

$$= x^3(16x^4-81) - (16x^4-81)$$
$$= (16x^4-81)(x^3-1) = (4x^2-9)(4x^2+9)(x^3-1)$$
$$= (2x+3)(2x-3)(4x^2+9)(x^3-1).$$

This example shows that sometimes we **must group certain terms** to get the difference of two squares.

EXERCISE XLII.

(*a*.)

Resolve into factors :—

1. $99^2 - 1$; $101^2 - 1$; $89^2 - 81^2$; $55^2 - 5^2$; $75^2 - 15^2$; $119^2 - 21^2$; $425^2 - 25^2$.

2. $x^2 - y^2$; $x^2 - 16$; $4x^2 - 9y^2$; $4x^2 - 1$; $1 - 9y^2$; $9a^2 - \frac{1}{4}b^2$.

3. $x^2 - 36y^2$; $4a^2 - z^2$; $18y^2 - 50z^2$; $16y^3 - a^2b^2y$; $4 - a^2b^2$; $100 - 4x^2$.

4. $9x^2 - y^2$; $a^2 - 16b^2$; $49m^4 - y^4$; $81 - x^4$; $a^2b^2 - 9c^2d^2$.

5. $4x^4 - y^4$; $1 - 4x^6$; $4x^8 - 1$; $16x^4 - y^4$; $9a^4x^8 - 1$.

6. $a^{2n} - b^{2n}$; $x^6y^6 - 4$; $\frac{1}{4}a^4 - 4$; $a^2b^4c^6 - x^6$; $1 - 4m^2x^{2n}$.

7. $15x^2y^2z^2 - 60a^2b^2c^2$; $243x^4 - 48y^4$; $3a^5b - 48ab^5$; $5 - 80a^4b^4c^4$.

(*b*.)

1. $(a + b)^2 - c^2$; $4(x + y)^2 - z^2$; $x^2 - (y + z)^2$; $4 - (a + b)^2$.

2. $(p + 2q)^2 - r^2$; $16x^2 - (a + 3b)^2$; $4m^2 - (p - q)^2$; $(x - 2y)^2 - (x + 2y)^2$.

3. $1 - (b - c)^2$; $(a + b + c)^2 - x^2$; $9 - (1 - x)^2$; $(b - c)^2 - (a - x)^2$.

4. $12(a^2 - bc)^2 - 3(b^2 - ac)^2$; $(a - 5b)^2 - 1$; $1 - (x - y + z)^2$; $a^8 - b^8$; $(a + 2b)^2 - (2b + 3c)^2$. $\longrightarrow (a - 3c)(a +$ [illegible]

5. $(a - 2b)^2 - (2a - 3b + 4c)^2$; $1 - (2a - a^2 + b^2)^2$; $(3 + 7x)^2 - (5x - 4)^2$.

6. $(x^2 - y^2 + z^2)^2 - 4x^2z^2$; $(x + y + z + u)^2 - (x - y + z - u)^2$; $(x^2 - y^2 - z^2)^2 - 4y^2z^2$.

7. $(x^2 - y^2 + z^2 - u^2)^2 - (2xz - 2yu)^2$; $a^2 - x^2 + 2xy - y^2$; $x^2 - y^2 - z^2 - 2yz$. $x+y-z+u)(x+z-y-u)(x-z+y-u)(x-z-y$

8. $x^2 - y^2 + 2yz - z^2$; $x^2 - y^2 - 2yz - z^2$; $x^2 - y^2 + 2xz + z^2$; $x^4 - x^2 + 2x \mp 1$.

$(a-c+b-d)(a-c-b+d)$

9. $x^2 + 2ax + a^2 - y^2 - 2yz - z^2$; $a^2 - 2ac - b^2 + 2bd + c^2 - d^2$; $a^{16} - b^{16}$; $(a + 2)^4 - (2a + 1)^2$. $a^2+6a+5)(a^2+2a+3)$

10. $a^2 - 2ab + b^2 - x^2 - 2xy - y^2$; $a^4 + 2a^3 + a^2 - b^4 + 2b^3 - b^2$.

11. $a^2b - bx^2 + a^2x - x^3$; $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$; $(b+x)(a+x)(a-x)$
$a^2b^2 + 2abc^2 - a^2c^2 - b^2c^2$. $(a-d+b-c)(a-d-b+c)$; $(a^2+ac-bc)(a^2-ac+bc$

12. $4a^2b^2 - (a^2 + b^2 - c^2)^2$; $x^4 + y^4 - z^4 + 2x^2y^2 - 2z^2 - 1$;
$a^2 - b^2 - c^2 + d^2 + 2bc - 2ad$. $(a+b+c)(a+b-c)(c+a-b)(c-a+b)$
$(x^2+y^2+z^2+1)(x^2+y^2-z^2-1)$; $(a-d+b-c)(a-d-b+c)$

13. $4a^2 - b^2 + 6bc - 9c^2$; $-4a^2 + 12ac + b^2 - 9c^2$;
$4a^2 - b^2 - 6bc - 9c^2$. $(2a+b-3c)(2a-b+3c)$
$(b+2a-3c)(b-2a+3c)$; $(2a+b+3c)(2a-b-3c)$

14. $-4a^2 + 4ab - b^2 + 9c^2$; $a^2 + 2ac - b^2 + c^2 - 2bd - d^2$;
$a^2 + 2ad - b^2 + d^2 - 2bc - c^2$. $(3c+2a-b)(3c-2a+b)$
$(a+c+b+d)(a+c-b-d)$; $(a+d+b+c)(a+d-b-c)$

15. $b^2 + 2bc - a^2 + c^2 - 2ad - d^2$; $a^2 + 2ad + d^2 - 4b^2 + 12bc - 9c^2$.

16. $9c^2 + 6cd + d^2 - a^2 + 4ab - 4b^2$; $a^2 - 6ac - 4b^2 + 9c^2 - d^2 + 4bd$.

17. $4(ad + bc)^2 - (a^2 - b^2 - c^2 + d^2)^2$; $(a^4 - 2a^2b^2 + b^4) - 4a^4b^4$.

18. $x^4 - \dfrac{1}{y^4}$; $x^8 - \dfrac{1}{256}$; $x^7 - 25x^3 + 6\tfrac{1}{4} - \tfrac{1}{4}x^4$; $x^7 - 16x^3 + x^4 - 16$.

104. By the application of the two preceding formulas, **expressions of the form** $x^4 + x^2y^2 + y^4$ may sometimes be resolved into factors. We **add** *some quantity to make the given expression a perfect square*, and **subtract** the *same quantity* from the square thus formed, so that the given expression remains *unchanged* in **value** though changed in **form**. Thus,

Ex. 1. $x^4 + x^2y^2 + y^4$ needs x^2y^2 to make it a square ; $\therefore$ adding and subtracting x^2y^2, the expression

$$\begin{aligned} &= x^4 + x^2y^2 + y^4 + x^2y^2 - x^2y^2 = x^4 + 2x^2y^2 + y^4 - x^2y^2 \\ &= (x^2 + y^2)^2 - x^2y^2 = (x^2 + y^2 + xy)(x^2 + y^2 - xy). \end{aligned}$$

Ex. 2. $$\begin{aligned} x^4 - 15x^2 + 9 &= x^4 - 6x^2 + 9 - 9x^2 \\ &= (x^2 - 3)^2 - 9x^2 \\ &= (x^2 - 3 + 3x)(x^2 - 3 - 3x). \end{aligned}$$

In this case we had merely to separate the middle term $-15x^2$ into two parts, viz., $-6x^2 - 9x^2$, etc.

Ex. 3. $$\begin{aligned} x^4 - 7x^2y^2 + y^4 &= x^4 - 7x^2y^2 + y^4 + 9x^2y^2 - 9x^2y^2 \\ &= x^4 + 2x^2y^2 + y^4 - 9x^2y^2 \\ &= (x^2 + y^2)^2 - 9x^2y^2 \\ &= (x^2 + y^2 + 3xy)(x^2 + y^2 - 3xy). \end{aligned}$$

Ex. 4. $4x^4 - 37x^2y^2 + 9y^4 = 4x^4 - 37x^2y^2 + 9y^4 + 12x^2y^2 - 12x^2y^2$
$= 4x^4 + 12x^2y^2 + 9y^4 - 49x^2y^2$
$= (2x^2 + 3y^2)^2 - 49x^2y^2$
$= (2x^2 + 3y^2 + 7xy)(2x^2 + 3y^2 - 7xy)$.

Ex. 5. Or the last Ex. may be resolved by *separating* the middle term $-37x^2y^2$ into $-12x^2y^2 - 25x^2y^2$; thus, Expression

$$= 4x^4 - 12x^2y^2 + 9y^4 - 25x^2y^2 = (2x^2 - 3y^2)^2 - 25x^2y^2$$
$$= (2x^2 - 3y^2 + 5xy)(2x^2 - 3y^2 - 5xy).$$

Ex. 6. $x^4 + 4y^4 = x^4 + 4y^4 + 4x^2y^2 - 4x^2y^2$
$= (x^2 + 2y^2)^2 - 4x^2y^2$
$= (x^2 + 2y^2 + 2xy)(x^2 + 2y^2 - 2xy)$.

Ex. 7. $a^4 + b^4 + (a+b)^4 = a^4 + b^4 + a^2b^2 + (a+b)^4 - a^2b^2$
$= (a^2 + b^2 + ab)(a^2 + b^2 - ab) + \{(a+b)^2 - ab\}\{(a+b)^2 + ab\}$
$= (a^2 + b^2 + ab)(a^2 + b^2 - ab) + \{a^2 + b^2 + ab\}\{a^2 + b^2 + 3ab\}$
$= (a^2 + b^2 + ab)(a^2 + b^2 - ab + a^2 + b^2 + 3ab)$
$= (a^2 + b^2 + ab)(2a^2 + 2ab + 2b^2) = 2(a^2 + ab + b^2)^2$.

Therefore to find the factors of such forms as $x^4 + x^2y^2 + y^4$:

1°. Take square roots of each *square* for *two* terms of the required factors; *i. e.* take roots of x^4 and y^4, *viz.* x^2 and y^2.

2°. From twice the product of these roots take the middle term (x^2y^2), and the square roots of the result will be the *third* terms of the required factors; *i. e.* $+xy$ and $-xy$.

EXERCISE XLIII.

Resolve into factors :—

1. $9x^4 + 5x^2y^2 + y^4$.
$3x^2 + y^2 \pm xy$
2. $16a^4 - 17a^2b^2 + b^4$. $(a-b)(a+b)$
$4a^2 - b^2 + 3ab\} \times (4a-b)(4a+b)$
3. $9a^4 - 19a^2b^2 + b^4$.
$3a^2 \mp b^2 \pm 5ab$
4. $25m^4 - 9m^2n^2 + 16n^4$.
$5m^2 + 4n^2 \pm 7mn$
5. $x^4 + x^2 + 1$. $x^2 + 1 \pm x$
6. $x^4 + 4x^2 + 16$. $x^2 + 4 \pm 2x$
7. $x^4 + 25x^2 + 625$. $x^2 + 25 \pm 5$
8. $a^4 + \frac{9}{4}a^2 + \frac{81}{16}$. $a^2 + \frac{9}{4} \pm \frac{3a}{2}$
9. $x^4 - 11x^2y^2 + y^4$. $x^2 - y^2 \pm 3x$
10. $x^4 + \frac{16}{9}x^2 + \frac{256}{81}$.
11. $a^4 + 6a^2y^2 + y^4$. $a^2 \quad y^2 \pm 2$
12. $m^4 + n^4 - 18m^2n^2$. $m^2 \quad n^2 \pm$

13. $81a^4 + 9a^2b^2 + b^4$.

$9a^2 + b^2 \pm 3ab$

14. $16a^4 + b^4 - 28a^2b^2$.

$4a^2 + b^2 \pm 6ab$

15. $25p^4 - 41p^2q^2 + 16q^4$.

$5p^2 - 4q^2 \pm pq$

16. $81x^4 - 34x^2y^2 + y^4$.

$9x^2 - y^2 \pm 4xy$

17. $4x^4 - 8x^2 + 1$. $2x^2 - 1 \pm 2x$

18. $\frac{1}{4}x^4 + y^4$. $\frac{1}{2}x^2 + y^2 \pm xy$

19. $x^4 + 4a^4y^4$. $x^2 + 2a^2y^2 \pm 2axy$

20. $4a^4 + y^4 - \frac{33}{4}a^2y^2$. $2a^2$ [illegible]

21. $x^8 + x^4y^4 + y^8$; $x^8 + x^4 + 1$. [illegible] x^2+1 [illegible]

$(x^4 + y^4 - x^2y^2)(x^2 + y^2 - xy)(x^2 + y^2 + xy)$

22. $a^4x^8 + a^2x^4 + 1$; $x^8 + 4y^8$.

$a^2x^4 + 1 \pm ax^2$; $x^4 + 2y^4 \pm 2x^2y^2$

23. $(a + b)^4 - 7c^2 (a + b)^2 + c^4$; $1 + 4x^4$.

$(a+b)^2 + c^2 \pm 3c(a+b)$; $1 + 2x^2 \pm 2x$

24. $16x^4 + 4 (y - z)^4 - 9x^2 (y - z)^2$; $1 + z^4 + 25z^8$.

$4x^2 + 2(y-z)^2 \pm 5x(y-z)$; $1 + 5z^4 \pm 3z^2$

25. $1 + 4a^8$; $a^4 + 81b^4 - 63a^2b^2$.

$1 + 2a^4 \pm 2a^2$; $a^2 + 9b^2 \pm$ [illegible]

26. $1 + a^4 + (1 + a)^4$; $x^4 + \frac{x^2}{y^2} + \frac{1}{y^4}$. [illegible]

$2(1+$ [illegible]

27. $x^4 + \frac{1}{4y^4}$; $a^4 + \frac{4}{a^4}$; $(a + b)^4 + (a - b)^4 + (a^2 - b^2)^2$.

$x^2 + \frac{1}{2y^2} \pm \frac{x}{y}$; $a^2 + \frac{2}{a^2} \pm 2$; $(a+b)^2 + (a-b)^2 \pm (a^2-b^2)$

28. $c^4 + 4 (a + b)^4$; $\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{a^2b^2}$; $\frac{9}{a^4} - \frac{3}{a^2b^2} + \frac{1}{b^4}$.

$c^2 + 2(a+b)^2 \pm 2c(a+b)$; $\frac{1}{a^2} + \frac{1}{b^2} \pm \frac{1}{ab}$; $\frac{3}{a^2} + \frac{1}{b^2} \pm \frac{3}{ab}$

105. Application of Formula (D).—

$$(x + a) (x + b) = x^2 + (a + b) x + ab.$$

Many **trinomials** may be resolved by this formula, which gives the product of two binomials that have the first terms *alike* and the second terms *unlike*.

Examine carefully how the products are formed in the following examples:

$(x + 7) (x + 5) = x^2 + (7 + 5) x + 7 \cdot 5 = x^2 + 12x + 35.$ (1)
$(x - 7) (x - 5) = x^2 - (7 + 5) x + (-7) (-5) = x^2 - 12x + 35.$ (2)
$(x + 7) (x - 5) = x^2 + (7 - 5) x + (+7) (-5) = x^2 + 2x - 35.$ (3)
$(x - 7) (x + 5) = x^2 - (7 - 5) x + (-7) (+5) = x^2 - 2x - 35.$ (4)

These examples represent all cases, and are included in the general form $x^2 \pm bx \pm c$, which will be referred to for the sake of brevity. Now, *referring to the required binomial factors:*

As to Terms.—1°. x^2 is the *square* of the *common* (or *like*) term in the required factors.

2°. $\pm b$ is the *sum* of two factors which will form $\pm c$.
3°. $\pm c$ is the product of the *second*, or *unlike*, terms.

As to Signs.—1°. $+c$ requires **both** *unlike* terms to have the **same** sign, **which must be the sign** of $\pm b$.

2°. $-c$ requires the *unlike* terms to have **different** signs, and the **greater term** *will have the sign of* $\pm b$.

Hence to *resolve the trinomial:*

1°. Take the square root of the *first* term (x^2) of the trinomial; this will be the *common* first term of each of the binomial factors.

2°. Resolve the *third term* into two factors, whose *sum* is the *second* term; these will be the *unlike* terms of the required factors.

Ex. 1. $x^2 + 12x + 35$.

1°. Here x will be the first term in each of the required factors.

2°. $+7$ and $+5$ are the Nos. whose *sum* is $+12$ and product $+35$; $\therefore$ the factors are $(x+7)(x+5)$.

Ex. 2. $x^2 - 12x + 35$.

1°. As in the last Ex., x will be the first term in each factor.

2°. The two Nos. whose product is $+35$ and *sum* -12, are -7 and -5; $\therefore$ the factors are $(x-7)(x-5)$.

Ex. 3. $x^2 + 2x - 35$.

1°. As before, the first term is x.

2°. The two Nos. whose product is -35 and *sum* $+2$, are $+7$ and -5; $\therefore$ the factors are $(x+7)(x-5)$.

Ex. 4. $x^2 - 2x - 35$.

1°. x is the first term in the required factors.

2°. The two Nos. whose product is -35 and *sum* -2, are -7 and $+5$; $\therefore$ the factors are $(x-7)(x+5)$.

Ex. 5. $x^6 - 33x^3 + 216$.

The factors of 216 are $9\cdot6\cdot4$; select a pair whose *sum* is -33; they are -24 and -9; $\therefore$ the factors are $(x^3-24)(x^3-9)$.

Ex. 6. $x^2 + 70x - 891$.

The factors of 891 are $9 \cdot 9 \cdot 11$; the pair 81 and 11 can be combined to make the coefficient 70, and the factors are $(x + 81)(x - 11)$.

Ex. 7. $x^3y - 7x^2y - 120xy = xy(x^2 - 7x - 120)$
$= xy(x - 15)(x + 8)$.

Ex. 8. $9x^2 - 6xy - 35y^2$: Here $3x$ is the *common* term, and ∴ we must resolve $-35y^2$ into two factors whose sum is $-2y$; these are $-7y$ and $+5y$ ∴ the factors are

$$(3x - 7y)(3x + 5y).$$

EXERCISE XLIV.

(a)

Resolve into factors :—

1. $x^2 + 5x + 6$.
2. $x^2 + 7x + 12$.
3. $x^2 + 9x + 20$.
4. $x^2 + 10x + 24$.
5. $x^2 + 18x + 72$.
6. $x^2 + 19x + 84$.
7. $x^2 + 17x + 60$.
8. $x^2 + 25x + 100$.
9. $x^2 + 11x + 24$..
10. $x^2 + 3x + 2$.
11. $x^4 + 7x^2 + 12$.
12. $a^2 + 22a + 105$.
13. $x^4 + 14x^2 + 49$.
14. $x^6 + 35x^3 + 150$.
15. $x^2y^2 + 17xy + 72$.
16. $a^2b^2c^2 + 49abc + 180$.
17. $m^6 + 40m^3 + 399$.
18. $a^4x^2 + 40a^2x + 39$.
19. $x^{2n} + 19x^n + 84$.
20. $x^2 + 40x + 391$.
21. $x^{2n} + 16x^n + 48$. [illegible]
22. $x^2 + 60x + 891$. $(x+27)(x+33)$
23. $a^4 + 40a^2 + 351$. $x^2+27)(x^2+13$
24. $a^4 + 36a^2b + 324b^2$.
    (a^2+18b) [illegible]
25. $a^4x^2 + 162a^2x + 6561$.
    (a^2x+81) [illegible]
26. $x^2 - 8x + 16$.
    $(x-4)^2$
27. $x^2 - 30x + 225$.
    $(x-15)^2$
28. $a^2 - 38a + 361$.
    $(a-19)^2$
29. $x^2 - 40x + 400$.
    $(x-20)^2$
30. $y^2 - 100y + 2500$.
    $(y-50)^2$
31. $y^6 - 30y^3 + 125$.
    $(y^3-5)(y^3-25)$
32. $m^2 - 22mn + 85n^2$.
    $(m-17n)(m-5n)$

33. $x^2 - 26xy + 169y^2$.
$x -$

34. $x^4 - 9x^2y^2 + 20y^4$.
$(x^2 - 5y^2)(x - 2y)(x + 2y)$

35. $a^2 - 29ab + 54b^2$.
$(a - 2b)(a - 27b)$

36. $12 - 7x + x^2$.
$(3 - x)(4 - x)$

37. $130 - 31ab + a^2b^2$.
$(26 - ab)(5 - ab)$

38. $a^2 - 40a + 375$.
$(a - 15)(a - 25)$

39. $x^4 - \frac{82}{9}x^2 + 1$. $x^2 - 9$ $x^2 - \frac{1}{9}$
$x - 3$ $x - 3$ $(x + \frac{1}{3})(x - \frac{1}{3})$

40. $x^6 - 35x^3 + 216$.
$(x^3 - 27)(x - 8)$

41. $3x^3 - 30x^2 + 48x$.
$3x(x - 8)(x - 2)$

42. $ax^2 - 11ax + 30a$.
$a(x - 5)(x - 6)$

43. $x^4 - 169x^2 + 3600$.
$(x - 12)(x + 12)(x - 5)(x +$

44. $x^{2n} - 44x^n + 259$. $(x^n - 7)(x^n - 37)$

45. $(a + b)^2 - 7(a + b) + 12$.
$(a + b - 4)(a + b - 3)$

46. $143 - 24ax + a^2x^2$.
$(13 - ax)(11 - ax)$

47. $1 - 59x^2y^2 + 408x^4y^4$.
$(1 - 51x^2y^2)(1 - 8x^2y^2)$

48. $a^2 - 54ab + 729b^2$.
$(a - 27b)(a - 27b)$

49. $a^2x^2 - 20abx^3 + 75b^2x^4$.
$x^2(a - 15bx)(a - 5bx)$

50. $m^2 - 38m + 361$.
$(m - 19)^2$

51. $p^2 - 54pq + 729q^2$. $(p - 27q)(p - 2$

52. $(x - y)^{2n} - 44(x - y)^n + 363$.
$\{(x - y)^n - 11\}\{(x - y)^n -$

(*b*)

1. $a^4 - a^2 - 2$. $(a^2 - 2)(a^2 + 1)$

2. $a^2 + a - 6$. $(a + 3)(a -$

3. $x^2 - x - 6$. $(x - 3)(x + 2)$

4. $x^2 - 13x - 48$. $(x - 16)(x + 3)$

5. $x^2 + 5x - 84$. $(x + 12)(x - 7)$

6. $y^2 + 7y - 60$. $(y + 12)(y - 5)$

7. $a^2 + 13a - 140$. $(a + 20)(a - 7)$

8. $a^2 + 13ab - 300b^2$.
$(a + 25b)(a - 12b)$

9. $x^2 + x - 132$.
$(x + 12)(x -$

10. $x^2 - 8x - 20$.
$(x - 10)(x + 2)$

11. $y^4 - 5a^2y^2 - 50a^4$.
$(y^2 - 10a^2)(y^2 + 5a^2)$

12. $a^2b^2 - 3ab - 4$.
$(ab - 4)(ab + 1)$

13. $3a^2z^4 - 39az^2 - 42$.
$3(az^2 - 14)(az^2 + 1)$

14. $a^8 - 15a^4 - 100$.
$(a^4 - 20)(a^4 + 5)$

15. $a^2b^2c^2 + 9abc - 22$.
$(abc + 11)(abc - 2)$

16. $a^4b^4 - 27a^2b^2 - 90$.
$(a^2b^2 - 30)(a^2b^2 + 3)$

17. $x^4 - 40x^2 - 384$.

18. $x^{2n} - 13x^n - 48$.

19. $(x + y)^2 - (x + y) - 342$.

20. $a^2 - 18(b + c)a - 360(b + c)^2$.

21. $x^{4n} + x^{2n} - 12$.

22. $380 - a - a^2$.

23. $65 + 8ab - a^2b^2$.

24. $204 - 5m - m^2$.

25. $3a^2y + 36abxy - 84b^2x^2y$.

26. $4x^2 + 24x + 35$.

27. $9x^2 + 36x + 35$. ?

28. $4x^4y^2 - 2x^2yz^2 - 42z^4$. $2x^2y -$

29. $49a^2 - 112ab + 64b^2$. $(7a - 8b)$

30. $4b^2x - 12bxy + 5xy^2$. $x($

31. $64x^4y^6 - 160x^4y^3z + 100x^4z^2$.

32. $a^4 - 35a^2b^2 - 200b^4$.

33. $121x^4 - 286x^2y + 169y^2$.

34. $9(a-b)^4 - 3c^2(a-b)^2 - 12c^4$.

35. $64a^{2n} - 8a^nb^n - 2b^{2n}$.

36. $\frac{1}{4}x^4 + \frac{1}{2}x^2 - 42$.

37. $\frac{9}{16}a^2 - \frac{3}{4}ab - 56b^2$.

38. $\dfrac{9}{x^2} + \dfrac{6}{x} - 63$.

39. $\dfrac{16}{x^2} - \dfrac{4}{x} - 380$.

40. $25x^4 - 50x^2y^2 - 651y^2$.

106. In the examples of the preceding article the trinomial to be resolved is formed from factors of the form $(mx+a)(mx+b)$ *i. e.* its *first term* is a *square* of the form m^2x^2. We can, however, extend the principle to resolve **any trinomial** *formed by binomial factors.* Let us see how such a trinomial is formed.

Multiply

$$\begin{array}{ll} & ax + by \\ \text{by} & \underline{cx + dy} \\ & acx^2 + bcxy \\ & \quad\;\; \underline{+ adxy + bdy^2} \\ & acx^2 + bcxy + adxy + bdy^2\,; \end{array}$$

in this it is plain that

$$acx^2 \cdot bdy^2 = bcxy \cdot adxy,$$

that is, *the product of the end terms = the product of the middle terms;* this is true also of the *coefficients.* Hence, to **factor any trinomial**, we first *resolve the product of the extreme end coefficients into two factors whose sum is equal to the coefficient of the middle term*, and then factor by parts (Art. 101).

Ex. 1 . $(3x+4)(2x+3) = 6x^2 + 8x + 9x + 12 = 6x^2 + 17x + 12$.

Now, conversely, it is required to resolve

$$6x^2 + 17x + 12.$$

Here the product of the extreme coefficients is $6 \times 12 = 72$; and the two factors of this, whose *sum* is 17, are 9 and 8 ; we have, therefore, the *expression*

$$\begin{aligned} &= 6x^2 + 9x + 8x + 12 \\ &= 3x(2x+3) + 4(2x+3) \\ &= (2x+3)(3x+4). \end{aligned}$$

Ex. 2 . $6x^2 - 13xy + 6y^2$: here 6×6 is to be resolved into two factors whose sum is -13 ; these are -9 and -4 : the *expression*, therefore,

$$\begin{aligned}&= 6x^2 - 9xy - 4xy + 6y^2\\&= 3x(2x - 3y) - 2y(2x - 3y)\\&= (2x - 3y)(3x - 2y).\end{aligned}$$

N. B.—If the last sign in the trinomial is +, the coefficient of the *middle* term immediately suggests the factors : *e. g.* in Ex. 1° and 2° we ask :—what *parts* of 17 will give the *product* 72 ? and what parts of 13 will give the *product* 36 ?

Ex. 3 . $15x^2 + 8xy - 12y^2$: here the product is 180, of which we must find the two factors whose sum is $+8$; these are 18 and 10 ; and since their sum is to be $+8$, we give 18 the sign +, and 10 the sign −. Hence the expression

$$\begin{aligned}&= 15x^2 + 18xy - 10xy - 12y^2\\&= 3x(5x + 6y) - 2y(5x + 6y)\\&= (5x + 6y)(3x - 2y).\end{aligned}$$

Ex. 4 . $28x^2 - 17x - 65$: here the product is $65 \times 28 = 13 \cdot 5 \times 4 \cdot 7$, of which the factors $52 (= 13 \cdot 4)$ and $35 (= 5 \cdot 7)$ can be combined to give -17 ; the sign of the larger must of course be −. Hence the expression

$$\begin{aligned}&= 28x^2 - 52x + 35x - 65\\&= 4x(7x - 13) + 5(7x - 13)\\&= (7x - 13)(4x + 5).\end{aligned}$$

Ex. 5 . $4(x + 2)^4 - 37x^2(x + 2)^2 + 9x^4$: for convenience write k for $(x + 2)^2$ $\therefore 4k^2 - 37kx^2 + 9x^4$: product $= 36$ which must be separated into two factors whose sum is -37 ; they are -1, -36. Hence we have the expression

$$\begin{aligned}&= 4k^2 - 36kx^2 - kx^2 + 9x^4\\&= 4k(k - 9x^2) - x^2(k - 9x^2)\\&= (k - 9x^2)(4k - x^2).\end{aligned}$$

Restoring the value of k and resolving by formula (B) we get

$$4(2x + 1)(1 - x)(3x + 4)(x + 4).$$

EXERCISE XLV.

1. $4x^2 + 8x + 3$. $(2x+1)(2x+3)$
2. $4x^2 + 13x + 3$. $(4x+1)(x+3$
3. $15x^2 + 27x + 12$. $3(x+1)(5x+4)$
4. $6x^2 + 7x + 2$. $(3x+2)(2x+1)$
5. $6x^2 + 23x + 20$. $(2x+5)(3x+4)$
6. $8x^2 + 34xy + 21y^2$. $(2x+7y)(4x+3y)$
7. $4a^2 + 13a + 9$. $(4a+9)(a+1)$
8. $7 + 10m + 3m^2$. $(1+m)(7+3m)$
9. $4x^2 + 23x + 15$. $(x+5)(4x+3)$
10. $3x^2 + 23x + 14$. $(x+7)(3x+2)$
11. $12x^2 - x - 6$. $(4x-3)(3x+2)$
12. $12x^2 + x - 6$. $(4x+3)(3x-2)$
13. $12x^2 + x - 35$. $(4x+7)(3x-5)$
14. $12x^2 - x - 35$. $(4x-7)(3x+5)$
15. $6x^2 + x - 2$. $(3x+2)(2x-1)$
16. $10x^2 - 17x + 3$. $(5x-1)(2x-3)$
17. $15x^2 + 14x - 8$. $(3x+4)(5x-2)$
18. $7x^2 - 50x + 7$. $(x-7)(7x-1$
19. $15x^2 - 19xy - 10y^2$.
$(5x+2y)(3x-5y)$
20. $a^4 - 2a^2 - 323$.
$(a^2-19)(a^2+17)$
21. $6m^2 - 17m - 380$.
$(3m+20)(2m-19)$
22. $6a^2 + 22a - 380$.
$2(a+10)(3a-19)$
23. $12x^2 + 13xy - 35y^2$.
$(3x+7y)(4x-5y)$
24. $15 - 27x - 132x^2$.
25. $20x^4 + x^2 - 1$.
$(5x^2-1)(4x^2+1)$
26. $15a^2 + 224a - 15$.
$(15a-1)(a+15)$
27. $24x^2 + 22x - 21$.
$(12x-7)(2x+3)$
28. $18 - 33y + 5y^2$.
$(6-y)(3-5y)$
29. $24a^2 - 29ab - 4b^2$.
$(8a+b)(3a-4b)$
30. $24 - 37y - 72y^2$.
$(8+9y)(3-8y)$
31. $28x^4 + 115x^2 - 125$
$(28x^2-25)(x^2+5)$
32. $56x^2 - 36xy - 20y^2$.
$4(14x+5y)(x-y)$
33. $56x^2 - 68xy + 20y^2$.
$4(7x-5y)(2x-y)$
34. $56x^2 + 36xy - 20y^2$.
$4(14x-5y)(x+y)$
35. $56a^2 - 67ab + 20b^2$.
$(8a-5b)(7a-4b)$
36. $56y^2 - 558y - 20$.
$2(28y+1)(y-10)$
37. $72y^2 - 19yz - 40z^2$.
$(8y+5z)(9y-8z)$
38. $36y^2 - 33ay - 15a^2$.
$3(3y+a)(4y-5a)$
39. $56a^4 - 276a^2b^2 - 20b^4$.
$4(14a^2+b^2)(a^2-5b^2)$
40. $56a^2 - 229ab + 20b^2$.
$(56a-5b)(a-4b)$
41. $39x^2 - 16xy - 48y^2$.
$(13x+12y)(3x-4y)$
42. $39x^2 - 20xy - 11y^2$.
$(3x+y)(13x-11y)$
43. $39x^2 + 13xy - 26y^2$
$13(3x-2y)(x+y)$
44. $60x^2 - 31xy - 104y^2$.
$(12x+13y)(5x-8y)$
45. $1 - 2x^2 - 143x^4$.
$(1-13x^2)(1+11x^2)$
46. $a^{2n} - 2b^na^n - 143b^{2n}$.
$(a^n-13b^n)(a^n+11b^n)$
47. $12x^6 - 51x^3 - 231$.
$3(x^3-7)(4x^3+11)$
48. $17x^2 + 288x - 17$.

107. Polynomials may sometimes be easily resolved into factors, when they contain **only one power of a certain letter**, by taking *all the terms containing this letter in one group*, and the remaining terms in a second group, and separating the *monomial* factors in each group. For such single power must be a term (or a coefficient) either in the multiplier *or* in the multiplicand, but NOT IN BOTH. If it occurs in the multiplier, it must be a *factor in the group of terms forming the multiplicand;* if it occurs in the multiplicand, it must be a *factor in the group of terms forming the multiplier;* in either case, the GROUP in which it is a factor is found by taking *all the terms that contain it;* and the other terms *must contain this* GROUP. Thus,—

Ex. 1. When x^2+ax+1 is multiplied by $ax+b$ we must have,

$$b \times (x^2 + ax + 1)$$

and

$$ax \times (x^2 + ax + 1).$$

Ex. 2. If $x^2 + ax + b$ is multiplied by $x + a$, we must have,

$$(x + a) \times (x^2 + ax)$$

and

$$(x + a) \times b.$$

Again,

Ex. 3. If $x^2 + ax + b$ is multiplied by $x + b$, we must have,

$$(x + b) \times ax$$

and

$$(x + b) \times (x^2 + b).$$

Now,

in Ex. 1. The group (x^2+ax+1) is got at once from the b terms.
" Ex. 2. " " $x+a$ " " " b "
" Ex. 3. " " $(x+b)$ " " " a "

Ex. 1 . Resolve $2x^2 - ax - 4bx + 2ab$. Here both a and b occur in only *one* power, we may therefore *group* the terms with respect to either letter. Take b. Then the expression

$$= \begin{cases} -4bx+2ab = -2b\,(2x-a) \\ +2x^2 - ax = \,x\,(2x-a) \end{cases}$$

$$\therefore \text{ the factors are } (2x-a)\,(x-2b).$$

Ex. 2 . $abx^2 - b^2x - a^2cx + abc$. Here c occurs in but *one* power, therefore take terms in c; thus,

$$\begin{aligned} c(-a^2x + ab) + (abx^2 - b^2x) &= -ac(ax - b) + bx(ax - b) \\ &= (ax - b)(bx - ac). \end{aligned}$$

Ex. 3 . $mx^3 - (m + a)x^2 + (a - am)x + a^2$; here only *one* power of m occurs, and we have,

i. $mx^3 - mx^2 - amx = mx(x^2 - x - a)$ . . . the m-terms grouped.
ii. $-ax^2 + ax + a^2 = -a(x^2 - x - a)$ . . . the rem'n'g terms grouped.

Hence the factors are $(mx - a)(x^2 - x - a)$.

Ex. 4 . $x^3 - (a - b)x^2 - (ab + 2b^2)x + 2ab^2$; here only *one* power of a occurs. Hence we have,

i. $-ax^2 - abx + 2ab^2 = -a(x^2 + bx - 2b^2)$ . . . . the a-terms grouped.
ii. $x^3 + bx^2 - 2b^2x = x(x^2 + bx - 2b^2)$ . . . . rem'g terms grouped.

And the factors are $(x - a)(x^2 + bx - 2b^2)$.

Ex. 5 . $x^3 - (a - b - 2)x^2 - (ab + 2a - 2b)x - 2ab$; here both a and b occur in only *one* power and we may take, as one group, either the a-terms or the b-terms. We have,

i. $-ax^2 - a(b+2)x - 2ab = -a\{x^2 + (b+2)x + 2b\}$ . . a-terms grouped
ii. $x^3 + (b+2)x^2 + 2bx = x\{x^2 + (b+2)x + 2b\}$ . . . rem. terms grouped.

The bracketed quantity is at once seen to have the factors $x + 2$ and $x + b$; hence the given quantity

$$\begin{aligned} &= (x - a)\{x^2 + (b + 2)x + 2b\} = (x - a)\{(x + b)(x + 2)\} \\ &\qquad = (x - a)(x + b)(x + 2). \end{aligned}$$

EXERCISE XLVI.

Resolve the following into factors:

1. $x^3 - (2 + a)x^2 + (2a - 1)x + a$. $(x-a)(x^2-$ [illegible]
2. $x^3 - (a + p)x^2 + (q + ap)x - aq$. $(x-a$ [illegible]
3. $may^3 + (mb - na)y^2 - (mc + nb)y + nc$. [illegible]
4. $(2b - c)x^2 + (2bc - 4b^2)x - bc + 2b^2$. [illegible]

[illegible]

5. $nx^3 - (n+a)x^2 - (n-a)x + a.$ $nx - a)(x^2 - x -$

6. $(m+1)(bx+an)b^2x^2 - (n+1)(mbx+a)a^2.$

7. $y^3 - (2a+b)y^2 + (2ab+a^2)y - a^2b.$ $y - b)(y - a)^2$

8. $x^3 - (a-b)x^2 - (ab+2b^2)x + 2ab^2.$ $(x-a)(x+2b)(x-b)$

9. $x^3 - (p^2+3q^2)x + 2p^2q - 2q^3.$ $x - 2q)(x + q + p)(x + q - p)$

10. $x^3 - (a-b-3)x^2 - (ab+3a-3b)x - 3ab.$ $(x+b)(x+3)(x-a$

× 11. $x^3 - (a-b+1)x^2 - (ab+b+a)x - ab.$ $x+b)(x^2 - ax - x - a$

12. $4x^3 - 2(a-4)x^2 - 2(2a+3)x + 3a.$ $(2x-a)(2x^2+4x-3)$

13. $py^3 - (p-q)y^2 + (p-q)y + q.$ $(py+q)(y^2-y+1)$

14. $mpx^3 + (mq-np)x^2 - (mr+nq)x + nr.$ $(mx-n)(px^2+qx-r)$

15. $max^3 - (mc+na)x^2 - (mb-nc)x + nb.$ $(mx-n)(ax^2-cx-b)$

16. $3px^3 - (pc+3q)x^2 - (pb-qc)x + qb.$ $(px-q)(3x^2-cx-b)$

17. $ax^4 - (ap-b)x^3 + (aq-bp-c)x^2 + (bq \pm cp)x - cq.$
$(x^2-px+q)(ax^2+bx-c$

← 18. $2x^3 + (2a+3c)x^2 + (3ac-4b)x - 6bc.$ $2x+3c)(x^2+ax-2b)$

19. $2x^3 - (4a-3c)x^2 + (6b-6ac)x + 9bc.$ $(2x+3c)(x^2-2ax+3b)$

20. $2ap^3 + (3a-2b)p^2q + (a-3b)pq^2 - bq^3.$ $(ap-bq)(2p^2+3$

← 21. $3ap^3 - (a+3b)p^2q + (2a-b)pq^2 + 2bq^3.$ $(ap-bq)(3p^2-pq-2$

22. $acx^3 + (bc+ad)x^2 + (bd+ac)x + bc.$ $(ax+b)(cx^2+dx+c)$

← 23. $2acx^3 + (2bc-ad)x^2 - (3ac+bd)x - 3bc.$ $(ax+b)(2cx^2-dx-3$

24. $27y^3 - 9(ab+bc)y^2 + 45y^2 + 3b(abc-5c-5a)y + 5ab^2c.$
$(3y-ab)(3y-bc)(3y-5)$

108. A polynomial of the second degree which has rational factors may be resolved by *taking its terms in trinomial groups*. Consider how such a polynomial is formed from its factors :—

Ex. 1.

$$
\begin{array}{l}
2x + y + 7z \\
x + 2y + 3z \\
\hline
2x^2 + xy + 7xz \\
\quad + 4xy + 2y^2 + 14yz \\
\qquad\qquad 6xz + 3yz + 21z^2 \\
\hline
2x^2 + 5xy + 2y^2 + 13xz + 17yz + 21z^2.
\end{array}
$$

This complete product is composed of the following parts :—

$$2x^2 + 5xy + 2y^2 = (2x + y)(x + 2y) \tag{1}$$
$$6xz + 7xz = 13xz = 2x\cdot 3z + x\cdot 7z \tag{2}$$
$$3yz + 14yz = 17yz = y\cdot 3z + 2y\cdot 7z \tag{3}$$
$$21z^2 = 3z\cdot 7z \tag{4}$$

Let us see how the original factors can be got from these parts,

(1) Gives directly $2x + y$
and $x + 2y$

the first *two* terms of the required factors, leaving $3z$ and $7z$, to be found.

(2) + (4) Gives $+ x\cdot 7z$
$+ 2x\cdot 3z + 21z^2$

which is clearly formed from the factors

$2x + 7z$
and $x + 3z$

the binomials got by *dropping the y-terms* of the original quantity. Hence, resolving $2x^2+13xz+21z^2$, we get $3z$ and $7z$, the required *third* terms. Hence, the factors of the given expression are $2x + y + 7z$ and $x + 2y + 3z$.

Observe:—These terms $3z$ and $7z$ may also be obtained from (3) + (4) that is, from the factors

$y + 7z$
and $2y + 3z$

the binomials got by dropping the x-terms of the original quantity.

Ex. 2. $6x^2 - 7xy + y^2 + 35xz - 5yz - 6z^2$. The first three terms give at once $6x - y$
and $x - y$.

Now drop the y-terms in the given expression and there results the trinomial

$$6x^2 + 35xz - 6z^2$$

the factors of which are $6x - z$
and $x + 6z$

hence $-z$ and $+6z$ are the required *third* terms, and the factors are $6x - y - z$ and $x - y + 6z$.

Ex. 3. $12x^2 - 13xy + 3y^2 - 4x + 13y - 56$. The first three terms give $3x - y$
and $4x - 3y$.

Drop the y-terms and we get $12x^2 - 4x - 56$, of which the factors are $3x - \mathbf{7}$
and $4x + \mathbf{8}$.

Hence, -7 and $+8$ are the required third terms and the given expression

$$= (3x - y - 7)(4x - 3y + 8).$$

We might have obtained -7, and $+8$, by dropping the x-terms in the given expression and resolving $3y^2 - 13y - 56$, observing, however, *that the signs of the factors of $3y^2$ must be* $-$ **as determined by the signs of the y's in $3x - y$ and $4x - 3y$.**

Hence, to factor such a polynomial having (say) x-terms, y-terms, and z-terms.

1°. Omit all z-terms, and factor the remaining group; this will give the first *two* terms of each of the required factors.

2°. Omit all y-terms and factor the remaining group; this will give the *third* terms of the required factors.

109. Another method of factoring such polynomials is to change the given expression into the form $a^2 - b^2$. We select the x-terms (say); multiply by 4 times the coefficient of x *square*, and by *adding* certain quantities form a COMPLETE SQUARE (see formula E.), and *if the remaining terms make a complete square*, we have the *difference* of two *squares*.

Ex. 1 . $2x^2 + 5xy + 2y^2 + 13xz + 17yz + 21z^2$: Multiply by 8 (4×2) then $16x^2 + 40xy + 16y^2 + 104xz + 136yz + 168z^2$; The *first* term of the square trinomial we need is $4x$, the second $5y$ (since twice $4x \times 5y = \mathbf{40}xy$), and $13z$ will give nearest square to $168z^2$ $\therefore$ we have

$$(4x + 5y + 13z)^2 - \mathbf{9}y^2 - z^2 + 6yz$$

(since we had to add $9y^2$ and z^2 to *complete* the trinomial square)

$$= (4x + 5y + 13z)^2 - (3y - z)^2$$
$$= (4x + 5y + 13z + 3y - z)(4x + 5y + 13z - 3y + z)$$
$$= (4x + 2y + 14z)(4x + 8y + 12z)$$

$= (2x + y + 7z)(x + 2y + 3z)$, by dividing the first factor by 2 and the second by 4, *i. e.* the whole by 8.

Ex. 2. $6a^2 - 7ab + 2ac - 20b^2 + 64bc - 48c^2$. Multiplying by 4×6, we have

$$144a^2 - 168ab + 48ac - 480b^2 + 1536bc - 1152c^2;$$

by *adding* $+ 49b^2 - 49b^2$ and $+ 4c^2 - 4c^2$ this becomes

$$= (12a - 7b + 2c)^2 - 529b^2 + 1564bc - 1156c^2$$
$$= (12a - 7b + 2c)^2 - (23b - 34c)^2$$
$$= (12a + 16b - 32c)(12a - 30b + 36c)$$

$= 24(3a + 4b - 8c)(2a - 5b + 6c)$: and dividing this by our multiplier, 24, we get the factors of the expression.

N. B.—If the coefficients of xy and xz are *both even*, we may complete the squares by multiplying by the coefficient of x^2 instead of 4 *times* that coefficient.

EXERCISE XLVII.

1. $6x^2 - 37xy + 6y^2 - 5x - 5y - 1$. $(6x - y + 1)(x - 6y - 1)$
2. $6x^2 - 5xy - 6y^2 - x - 5y - 1$. $(3x + 2y + 1)(2x - 3y - 1)$
3. $12a^2 - ab - 20b^2 - 8a - 41b - 20$. $(3a - 4b - 5)(4a + 5b + 4$
4. $x^2 + xy - 2y^2 + 2xz + 7yz - 3z^2$. $(x - y + 3z)(x + 2y - z)$
5. $3x^2 - 8xy - 3y^2 + 30x + 27$. $(3x + y + 3)(x - 3y + 9)$
6. $6a^2 - 7ab + 2ac - 20b^2 + 64bc - 48c^2$. $(3a + 4b - 8c)(2a - 5b + 6c$
7. $12a^2 - 13ab - 4a + 3b^2 + 13b - 56$. $(4a - 3b + 8)(3a - b - 7)$
8. $7x^2 - 8xy + y^2 + 20x - 2y - 3$. $(7x - y - 1)(x - y + 3)$
9. $a^2 - ay - 12y^2 - 5a - 15y$. $(a + 3y)(a - 4y - 5)$
10. $4x^2 - 15y^2 - 4xy - 21z^2 - 36yz - 8xz$.
$(2x - 5y - 7z)(2x + 3y + 3z)$
11. $9x^2 - 6xy - 3y^2 + 8z^2 - 18xz + 10yz$.
$(3x + y - 4z)(3x - 3y - 2z)$
12. $6x^2 - 13xy + 12xz + 6y^2 - 13yz + 6z^2$.
$(3x - 2y + 3z)(2x - 3y + 2z)$
13. $5x^2 - 8xy + 3y^2 - 3xz + yz - 2z^2$.
$(5x - 3y + 2z)(x - y - z)$
14. $14a^2 - 29ab + 41ac - bc + 2b^2 - 3c^2$.
$(14a - b - c)(a - 2b + 3c)$
15. $8a^2 - 10ab - 14ac + 10bc + 3b^2 + 3c^2$.
$(4a - 3b - c)(2a - b - 3c)$
16. $1 + 4x - y - 20y^2 - 21x^2 + 43xy$.
$(1 + 7x - 5y)(1 - 3x + 4y)$

110. Exact Division.—We can sometimes discover by trial whether one quantity is a factor (*i. e.*, an *exact* divisor) of another. The principle on which the process depends is a very simple one. In the explanations a stands for any *one-dimension* quantity, as $a + y$, $a + b$, $a + b + c$, $x + y + z$, etc.

1°. A quantity *vanishes* (*i. e.*, *becomes zero*) when one of its factors, a, *vanishes* (*i. e.*, *becomes zero*).

2°. Conversely, if a quantity *becomes zero* when a becomes zero, then *a is a factor* of that quantity.

For example, ax, a^2x^2, $a(x + y)$, $ax - ay^2 + a^2z^3$, etc., each equal zero when $a = 0$. So,

$$b(x - a),\ x^2 - a^2,\ (x - a)(a + b + c),\ \text{etc.},$$

become zero when $x - a = 0$.

On the other hand

$$ax + a^2y^2 + b^2y^2,\ \ ax^2 + bx + a,$$

do *not* vanish when $a = 0$, because they do not contain a as a factor. So,

$$a^2(x - a) + b^2(x^2 - a^2) + c^2(x^2 + a^2)$$

does *not* vanish when $x - a = 0$, since $x - a$ is not a factor of it.

3°. If a quantity does not vanish when $a = 0$, the result is the *remainder which would be obtained by dividing the quantity by a*. Thus,

put $a = 0$ in $ax + a^2y^2 + b^2y^2$, and there results

$$0 + 0 + b^2y^2 = b^2y^2,$$

which is the *remainder obtained by dividing* the same quantity by a. So, in

$$a^2(x - a) + b^2(x^2 - a^2) + c^2(x^2 + a^2),$$

if we put $\quad x - a = 0,\ i.\ e.\ x = a,$

we get $\quad 0 + 0 + c^2(a^2 + a^2) = 2a^2c^2,$

which is the remainder obtained by dividing the given expression by $x - a$.

Ex. 1. Is $x-3$ a factor of x^2+x-12?

Put $$x-3=0, \quad \therefore x=3.$$

Substitute 3 for x in the expression; then it becomes

$$3^2+3-12=0,$$

$\therefore x-3$ is a factor.

Ex. 2. Is $x+3$ a factor of $x^3-2x^2-10x+15$?

Let $$x+3=0, \quad \therefore x=-3.$$

Substitute -3 for x; then the expression becomes

$$\begin{aligned}&(-3)^3-2(-3)^2-10(-3)+15\\ =\ &-27-18+30+15=0;\end{aligned}$$

$\therefore x+3$ is a factor.

Ex. 3. Find whether $a+b-c$ is a factor of $a^3+b^3-c^3+3abc$.

Let $$a+b-c=0, \quad \therefore c=a+b.$$

Substitute this value of c in the given expression, and it becomes

$$\begin{aligned}&a^3+b^3-(a+b)^3+3ab(a+b)\\ =\ &(a+b)^3-(a+b)^3=0.\end{aligned}$$

$\therefore a+b-c$ is a factor.

Ex. 4. Find the remainder when x^3-3x^2+2x-7 is divided by $x-2$.

Let $$x-2=0, \quad \therefore x=2.$$

Substitute this value of x; then the expression becomes

$$\begin{aligned}&2^3-3\cdot 2^2+2\cdot 2-7\\ =\ &8-12+4-7=-7.\end{aligned}$$

Ex. 5. What is the remainder on dividing $x^2-(a+1)x+2a-1$ by $x-a+1$?

Let $$x-a+1=0, \quad \therefore x=a-1.$$

Substitute this for x; then the expression

$$\begin{aligned}&=(a-1)^2-(a+1)(a-1)+2a-1\\ &=a^2-2a+1-(a^2-1)+2a-1=1.\end{aligned}$$

Ex. 6. Show that $(b + c - a)(c + a - b)(a + b - c) + 8abc$ is exactly divisible by $a + b + c$.

Let $a + b + c = 0$, $\therefore c = -(a + b)$.

Substituting this value for c, the expression

$$\begin{aligned} &= \{b-(a+b)-a\}\{-(a+b)+a-b\}\{a+b+(a+b)\}+8ab\times-(a+b) \\ &= \{\quad -2a \quad\}\{\quad -2b \quad\}\{\quad 2(a+b) \quad\}-8ab(a+b) \\ &= \qquad 8ab(a+b) - 8ab(a+b) = 0. \end{aligned}$$

The method of synthetic division is useful in finding *remainders* and *values*, especially when the divisor is a binomial, $x \pm a$, for example.

Ex. 7. Find the value (or remainder) of

$x^5 + 5x^4 + 10x^3 + 1$[illegible]$x^2 + 5x + 1$ when $x = -1$, or $x + 1 = 0$;

divide by $x + 1$,

$$\begin{array}{r|l} 1 & 1 + 5 + 10 + 10 + 5 + 1 \\ -1 & \quad - 1 - \ 4 - \ 6 - 4 - 1 \\ \hline & 1 + 4 + \ 6 + \ 4 + 1; + 0 \end{array}$$

$\therefore$ the *remainder* is zero when the quantity is divided by $x + 1$; or, the *value* is zero when $x = -1$.

Ex. 8. Find the value of $8x^3 + 12x^2 - 4x - 5$ when $x = -\frac{3}{2}$.

If $x = -\frac{3}{2}$, $\therefore x + \frac{3}{2} = 0$, or $2x + 3 = 0$;

divide by this,

$$\begin{array}{r|l} 2 & 8 + 12 - 4 - 5 \\ -3 & 4 - 12 + 0 + 6 \\ \hline & 4 + \ 0 - 2; + 1 \end{array} \qquad \therefore 1 \text{ is the value.}$$

Ex. 9. Show that $x^n - a^n$ is divisible by $x - a$.

Let $x - a = 0$, $\therefore x = +a$; and, substituting this value, the expression

$$= (+a)^n - a^n = +a^n - a^n = 0.$$

Try also whether $x + a$ is a factor.

Let $x + a = 0$. $\therefore x = -a$; and the expression

$$= (-a)^n - a^n = +a^n - a^n = 0, \quad \text{when } n \text{ is } \textit{even}; \text{ but}$$
$$= -a^n - a^n = -2a^n, \text{ when } n \text{ is } \textit{odd}.$$

Ex. 10. Show that $x^n + a^n$ is divisible by $x + a$ when n is *odd*, but *not* when n is *even*.

Put $x + a = 0$, $\therefore x = -a$; then the expression
$\doteq (-a)^n + a^n = -a^n + a^n \doteq 0$, when n is *odd*; but
$= +a^n + a^n = 2a^n$, when n is *even*.

EXERCISE XLVIII.

1. Show that $x^2 - x - 6$ is divisible by $x - 3$ and by $x + 2$.

-14 2. Show that $2x^2 + x - 28$ is divisible by $x + 4$ and by $2x - 7$.

3. Show that $2x^3 + 3x^2 - 98x - 147$ is divisible by $x \pm 7$.

4. Prove that $x^3 - (a + 2)x^2 + (1 + 2a)x - a$ is divisible by $x - a$.

5. Prove that $x^3 - 2ax^2 + (a^2 + b)x - ab$ is divisible by $x - a$.

6. Prove that $x^3 + (2a + b)x^2 + (a^2 + 2ab)x + a^2b$ is divisible by $x + a$.

Find the remainders in the following cases :—

x 7. When $x^5 - 7x^4 + 13x^3 - 16x^2 + 9x - 12$ is divided by $x - 5$.

30 8. When $(y - a)^3 + (y - b)^3 + (a + b)^3$ is divided by $y + a$.

-785 9. When $8x^3 - 16x^2 - 12x - 10$ is divided by $2x + 5$.

/ 10. When $x^5 - 98x^4 - 98x^3 - 100x^2 + 98x + 100$ is divided by $x - 99$.

11. When $x^3 + px^2 + qx + r$ is divided by $x - a$. $a^3 + pa^2 + qa + r$

12. When $x^5 - 3x^4 + 5x^2 - 40$ is divided by $x - 2$. -3[illegible]

13. When $x^4 - x^3 + x^2 - x + 1$ is divided by $x + 6$. 1555

14. Show that $x - 4$ is a factor of $x^3 - 5x^2 - 2x + 24$; and find the other factors. $(x+2)(x-3)(x-4)$

15. Find the factors of $6x^3 + 7x^2 - x - 2$. $(x+1)(3x+2)(2x-1)$

16. Show that $x^3 - 7x^2a + 8xa^2 + 52a^{\overset{3}{2}}$ is divisible by $x + 2a$.

17. Show that $x^4 + 4x^3 - 5x^2 - 36x - 36$ is divisible by $x^2 - x - 6$.

18. Find the factors of $x^3 - 39x + 70$.

Find the value of:—

19. $x^5 - 19x^2 - 121$, when $x = -3$. -535

20. $x^5 - 4x^3 - 2x^2$, when $x = -4$. -800

21. $x^6 - 102x^5 + 100x^4 + 102x^3 - 99x^2 - 201x$, when $x - 101 = 0$. / /

22. $3x^4 + 60x^3 + 54x^2 - 60x + 58$, when $x = -19$. 115

23. What quantity must be subtracted from $x^3 - px^2 + qx - r$ to give a remainder exactly divisible by $x + a$? [illegible]

24. Find the remainder when $x^3 + a^3 + b^3 + (x + a)(x + b)(x + c)$ is divided by $x + a + b$. [illegible]

✓ 25. Show that $ay^3 - (a^2 + b)y^2 + b^2$ is divisible by $ay - b$.

✓ 26. Show that $x(x + 2y)^5 - y(y + 2x)^5$ is divisible by $x \pm y$.

✓ 27. Prove that $a^3 + b^3 + c^3 - 3abc$ is divisible by $a + b + c$.

✓ 28. Prove that $a^3(b - c) + b^3(c - a) + c^3(a - b)$ is divisible by $a + b + c$.

29. Find the value of $10y^4 - 1109y^3 - 109y^2 - 212y - 1111$, when $y = 111$. -1

30. Prove that $a^2x^3 + a^5 - 2abx^3 + b^2x^3 + a^3b^2 - 2a^4b$ is divisible by $(a - b)(x + a)$.

✓ 31. Show that $(a - b)(x - a)(x - b) + (b - c)(x - b)(x - c) + (c - a)(x - c)(x - a)$ is divisible by $a - b$, $b - c$, and $c - a$.

32. What number added to $4x^5 + 34x^4 + 58x^3 + 21x^2 - 123x - 41$ will give a result divisible by $2x + 13$? × 2

✓ 33. Show that $a^2(b^2 + c^2 - a^2) + b^2(c^2 + a^2 - b^2) + c^2(a^2 + b^2 - c^2)$ is divisible by $a + b + c$.

34. What does $a^3 + b^3 + c^3 + 3abc$ become, when $a + b - c = 0$? $2c^3$

✓ 35. What does $a^3 - b^3 - c^3 - 3abc$ become, when $a - b - c = 0$? 0

36. Show that $b(x^3 + a^3) + ax(x^2 - a^2) + a^3(x + a)$ is divisible by $(x + a)(a + b)$.

37. Prove that $(x + y + z)(xy + yz + zx) - xyz$ has $x + y$, $y + z$, and $z + x$ for factors.

38. Prove that $a^3 - b^3 + c^3 + 3abc$ is zero, if $a - b + c = 0$. ×

39. What does $ax^3 - (a^2 + b)x^2 + b^2$ become, when $ax + b = 0$?

40. What does $(x + y)(y + z)(z + x) + xyz$ become, when $x + y + z = 0$?

41. Is $x + y - 1$ a divisor of $(x + y)^3 + 3xy(1 - x - y) - 1$?

42. What is the value of $x^3 + (4ab - b^2)x - (a - 2b)(a^2 + 3b^2)$, when $x - a + 2b = 0$? $a^3 - 2a^2b + 3ab^2 - 6b^3$

✓ 43. Show that $(a - b)x^3 - (x - b)a^3 + (x - a)b^3$ is divisible by $a - b$ and by $x + a + b$.

44. What must be subtracted from each of the quantities $x^2 - px + q$ and $x^2 - p'x + q'$, in order that both results may be divisible by $x + a$?

✓ 45. Show that $x^4 - px^3 + qx^2 - rx + s$ is exactly divisible by $x - a$, if $a^4 - pa^3 + qa^2 - ra + s = 0$.

111. Certain cases of exact division are of special importance, and the results should be carefully memorized. These are $(x^n \pm a^n) \div (x \pm a)$. We find at once, by the last article, that

1°. $x^n - a^n$ is divisible by $x - a$, **always.**

2°. $x^n - a^n$ is divisible by $x + a$, **when n is even.**

3°. $x^n + a^n$ is divisible by $x + a$, **when n is odd.**

These are the only cases in which the *sum* or *difference* of *like powers* of two quantities is divisible by the *sum* or *difference* of the *quantities.*

Ex. 1°. $\dfrac{x^6 - a^6}{x - a} = x^5 + x^4a + x^3a^2 + x^2a^3 + xa^4 + a^5$; and, *generally*,

$$\frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + xa^{n-2} + a^{n-1}. \quad \text{(I.)}$$

Ex. 2°. $\dfrac{x^6 - a^6}{x + a} = x^5 - x^4a + x^3a^2 - x^2a^3 + xa^4 - a^5$; and, *generally*,

$$n \text{ even}, \quad \frac{x^n - a^n}{x + a} = x^{n-1} - x^{n-2}a + x^{n-3}a^2 - \dots + xa^{n-2} - a^{n-1}. \quad \text{(II.)}$$

Ex. 3°. $\dfrac{x^7 + a^7}{x + a} = x^6 - x^5a + x^4a^2 - x^3a^3 + x^2a^4 - xa^5 + a^6$; and, *generally*,

n odd, $$\frac{x^n + y^n}{x + y} = x^{n-1} - x^{n-2}a + x^{n-3}a^2 - \ldots - xa^{n-2} + a^{n-1}. \quad \text{(III.)}$$

N. B.—The student is once more reminded that the letters in these and other algebraic formulas denote *any quantities.*

Ex. 1. $a^3 - (b + c)^3 \div a - (b + c)$, formula (I.),

$$= a^2 + a(b + c) + (b + c)^2.$$

Ex. 2. $x^4 - 81 \div x + 3$. This is

$$\frac{x^4 - (3)^4}{x + 3} = x^3 - x^2 \cdot 3 + x \cdot (3)^2 - (3)^3$$
$$= x^3 - 3x^2 + 9x - 27, \text{ formula (II.)}.$$

Ex. 3. What factor multiplied into $x^4 - x^3y + x^2y^2 - xy^3 + y^4$ will give $x^5 + y^5$?

We find $$\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4, \text{ formula (III.)};$$
hence $x + y$ is the required factor.

Ex. 4. Show that $(a + b - 2c)^3 + (a - b - 2c)^3$ is exactly divisible by $2(a - 2c)$.

The expression is the *sum* of *odd* powers, and is therefore (formula III.) divisible by the sum of the quantities, *i. e.*, by $(a + b - 2c) + (a - b - 2c)$ which $= 2a - 4c = 2(a - 2c)$.

Ex. 5. Resolve into factors $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$.

By formula (I.) we see at once that this is

$$\frac{a^6 - b^6}{a - b} = \frac{(a^3 - b^3)(a^3 + b^3)}{a - b}$$
$$= (a^2 + ab + b^2)(a^3 + b^3)$$
$$= (a^2 + ab + b^2)(a + b)(a^2 - ab + b^2).$$

Ex. 6. Prove that $\{6(x-1)\}^7 + (2x^2 + 2x - 4)^7 - \{2(x^2 + 4x - 5)\}^7$ is exactly divisible by $2x - 2$ and by $x + 5$.

The first two expressions are divisible by

$$6(x - 1) + 2x^2 + 2x - 4$$
$$= 2x^2 + 8x - 10 = 2(x^2 + 4x - 5),$$

which divides the third quantity, and has the factors

$$2(x-1)(x+5).$$

Ex. 7. $(x^{12}+a^{12}) \div (x^4+a^4)$; here we have

$$\frac{(x^4)^3+(a^4)^3}{x^4+a^4} = (x^4)^2 - x^4a^4 + (a^4)^2 = x^8 - x^4a^4 + a^8.$$

Ex. 8. $\left[\left(\frac{ab}{cd}\right)^4 - \left(\frac{cd}{ab}\right)^4\right] \div \frac{a^2b^2+c^2d^2}{abcd}$; the divisor $= \frac{ab}{cd} + \frac{cd}{ab}$;

$$\therefore \text{ the quotient} = \left(\frac{ab}{cd}\right)^3 - \left(\frac{ab}{cd}\right)^2 \cdot \frac{cd}{ab} + \frac{ab}{cd} \cdot \left(\frac{cd}{ab}\right)^2 - \left(\frac{cd}{ab}\right)^3.$$

EXERCISE XLIX.

Divide :—

1. $x^8 - 1$ by $x - 1$; $x^6 - a^6$ by $x + a$; $x^5 + (2y)^5$ by $x + 2y$.
2. $x^7 + 1$ by $1 + x$; $(ax)^7 - 1$ by $ax - 1$; $1 - \left(\frac{a}{b}\right)^4$ by $1 + \left(\frac{a}{b}\right)$.
3. $(a+b)^3 - 1$ by $a + b - 1$; $x^6 + a^3$ by $x^2 + a$.

$(a+b)^2 + (a+b) + 1$

4. $\left(\frac{x}{a}\right)^4 - \left(\frac{a}{x}\right)^4$ by $\frac{x}{a} - \frac{a}{x}$; $(1-x)^3 + (1+3x)^3$ by $2 + 2x$. $1 + 2x + 13x$

$\frac{x}{a}$ + [illegible]

5. $(p-1)^3 + (q+1)^3$ by $p + q$; $a^{2n} - b^{2n}$ by $a^n + b^n$.

$p-1$ [illegible] $(q+1)^2$

6. $x^{15} + a^{15}$ by $x^5 + a^5$; $1 - (a-b)^4$ by $1 + a - b$.

$x^{10} - x^5a^5 + a^{10}$ $1 - (a-b) + (a-b)^2 - (a-b)^3$

7. $x^3 + \frac{1}{x^3}$ by $x + \frac{1}{x}$; $x^{12} + 2x^6 + 1$ by $x^4 + 2x^2 + 1$. x [illegible]

$x^2 - 1 + \frac{1}{x^2}$

Resolve into factors :—

8. $x^4 + x^2y^2 + y^4$; $a^4 - 16b^4$; $(x-2y)^3 + (x+2y)^3$. $x($ [illegible] $)$

$(x^2 \pm xy - y^2)$ $(a^2+4b^2)(a+2b)(a-2b)$

9. $a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5$; $x^{10} - 32y^{10}$.

$(a+b)(a^2+b^2 \pm ab)$ $(x^2-2y^2)($ [illegible] x^4 [illegible] $8x^2$ [illegible] $16y^8)$

10. $(a^2-bc)^3 + 8b^3c^3$; $(x^2-1)^3$; $(x^4-1)(x^4+1)$.

$(a^2+bc)(a^4+$ [illegible] $-4a^2bc)$; $(x+1)^3(x-1)^3$; $(x-1)(x+1)(x^2+1)$ [illegible]

11. $a^3 - 8b^{15}$; $a^3 - b^3 - a(a^2-b^2) + b(a-b)^2$.

$(a-2b^5)(a^2+2ab^5+4b^{10})$; a [illegible] $(a-b)$

12. $4a^2(x^3+18ab^2) - (32a^5+9b^2x^3)$; $a^3 - \frac{1}{8a^3}$. $a - \frac{1}{2a}$ [illegible] $+\frac{1}{2}$ [illegible]

$(x-2a)(x^2+2ax+4a^2)(x+3b)(x-3b)$

13. Show that $(3x^2-7x+2)^3 - (x^2-8x+8)^3$ is exactly divisible by $2x^2 + x - 6$.

14. Show that $(7a^2+3a-3)^3+(5a^2-4a-3)^3$ is divisible by $4a-3$, and by $3a+2$.

15. Find the value of $x^4-2x^3y+2xy^3-y^4$, when $x=(a+b)^2$ and $y=(a-b)^2$. $125\,a^2b^3(a^2+b^2)$

16. Divide x^9-1 by the product of x^2+x+1 and x^6+x^3+1. $x-1$

17. Divide $a^{16}+x^8a^8+x^{16}$ by the product of a^2-ax+x^2 and a^2+ax+x^2. [illegible]

18. Find the six factors of $9x^6y^2-576y^2-4x^8+256x^2$.

19 If $a^3+\dfrac{1}{a^3}=0$, show that $\left(a+\dfrac{1}{a}\right)^2=3$.

20. Show that $1-x-x^n+x^{n+1}$ is divisible by $1-2x+x^2$.

21. Show that $(x^2-2x+1)^6-x^6+(x^2-x+1)^2$ is divisible by x^3-2x^2+2x-1.

22. Write down the quotient of $x^7-x^6y+x^5y^2-x^4y^3+x^3y^4-x^2y^5+xy^6-y^7$ by x^4+y^4. $x^3-x^2y+xy^2-y^3$

112. When one factor of an expression, which is symmetric with respect to two or more letters, has been found, the principle of symmetry may be applied to determine the other factors.

Ex. 1. To factor $(x+y+z)(xy+yz+zx)-(x+y)(y+z)(z+x)$.

1°. Observe that the expression is symmetrical in x, y, z, and is of *three dimensions*; $\therefore$ whatever is found to be true of x (say) is true also of y and z.

2°. To see if there is a monomial factor put $x=0$ in the expression, and it becomes $(y+z)(0+yz+0)-(0+y)(y+z)(z+0)$

$$=y^2z+yz^2 \qquad -(y^2z+yz^2)=0.$$

$\therefore x$ is a factor; and by symmetry, y and z are known to be factors, and $\therefore xyz$ is a factor.

3°. There can be no other *literal* factor, for xyz is of three dimensions, but there may be a *numerical* factor, n suppose, so that $(x+y+z)(xy+yz+zx)-(x+y)(y+z)(z+x)=nxyz$, which is true for all values of x, y, and z: assume $x=y=z=1$, and substitute; $\therefore(1+1+1)(1+1+1)-(1+1)(1+1)(1+1)=n\cdot1\cdot1\cdot1$

or $\qquad 9 \quad - \quad 8 \quad = \quad n\cdot 1$

$\therefore n=1$, and x, y, z, are the factors.

Ex. 2. $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$.

1°. This expression is symmetrical and of three dimensions.

2°. It does not $= 0$ when $a = 0$ $\therefore$ there is no monomial factor.

3°. Put $a - b = 0$ *i. e.* $a = b$, and substitute ; $\therefore$ the expression becomes

$$a(a^2 - c^2) + a(c^2 - a^2) + c(a^2 - a^2)$$
$$= \quad a^3 - ac^2 + ac^2 - a^3 = 0$$

$\therefore a - b$ is a factor, and therefore $b - c$, $c - a$ are also factors : $\therefore$ their product $(a - b)(b - c)(c - a)$ is a factor.

4°. There can be no other *literal* factor ; but there may be a *numerical* factor ; let n denote this ; so that,

$$a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2) = n(a - b)(b - c)(c - a).$$

To find n put $a = 1$, $b = 2$, $c = 0$, and this expression becomes

$$1(4 - 0) + 2(0 - 1) + 0 = n(1 - 2)(2 - 0)(0 - 1)$$
$$\text{or} \qquad 4 - 2 = n(-1)\,2\,(-1)$$
$$\therefore \qquad 2 = 2n, \ \therefore n = 1.$$

Ex. 3. $a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2$.

This is symmetrical in a, b, c, and is of *four* dimensions ; the double product suggests that it is formed by *trinomial* factors. Put $a + b - c = 0$ *i. e.*, $c = a + b$ in the expression, and it becomes zero ; $\therefore a + b - c$ is a factor, and by symmetry $b + c - a$. and $c + a - b$ are factors and $\therefore$ $(a + b - c)(b + c - a)(c + a - b)$ is a factor ; this is of *three* dimensions, but the given expression is of *four* dimensions : there is therefore *another literal factor* to be found ; to find this factor, we know,

1°. It must contain all three letters a. b, c.

2°. It must be symmetric in these letters.

3°. It must be of *one* dimension.

The *only* factor that satisfies these conditions is $a + b + c$; we have therefore

$$a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2$$
$$= n(a + b + c)(a + b - c)(b + c - a)(c + a - b).$$

Put $a = b = c = 1$, and we get $n =$ ~~1.~~ -1

EXERCISE L.

(*a*)

Resolve :—

1. $(x + y + z)^3 - (x^3 + y^3 + z^3)$. $3(x+y)(y+z)(z+x)$
2. $a^2(b - c) + b^2(c - a) + c^2(a - b)$. $-(a-b)(b-c)(c-a)$
3. $bc(b - c) + ca(c - a) + ab(a - b)$. (a − b)(b − c)([illegible] − c)
4. $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$. (a − b)(b − c)(c − a)
5. $(x - y)^3 + (y - z)^3 + (z - x)^3$. $3(x-y)(y-z)(z-x)$
6. $(a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3$. $3(a^2-b^2)(b^2-c^2)(c^2-a^2)$
7. $(x + y + z)(xy + yz + zx) - xyz$. (x + y)(y + z)(z + x)
8. ✓ $a^3(b - c) + b^3(c - a) + c^3(a - b)$. (a − b)(b − c)(c − a)(a + b + c)
9. $a(b - c)^3 + b(c - a)^3 + c(a - b)^3$. (a − b)(b − c)(c − a)(a + b + c)
10. $x^2(y + z) + y^2(z + x) + z^2(x + y) + 2xyz$. $(x+y)(y+z)(z+x)$
11. $a^4(c - b^3) + b^4(a - c^3) + c^4(b - a^3) + abc(a^2b^2c^2 - 1)$. $-(a-c^3)(b-a^3)(c-b^3)$
12. $a^4(b - c) + b^4(c - a) + c^4(a - b)$. $-(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+bc+ca)$
13. $ab(a + b) + bc(b + c) + ca(c + a) + a^3 \div b^3 \div c^3$. $(a+b+c)(a^2+b^2+c^2) -$
14. $x^3 + y^3 + z^3 + 3(x + y)(y + z)(z + x)$. $(x+y+z)^3$
15. $a^3 + b^3 + c^3 + 3(a + b + c)(ab + bc + ca) - 3abc$. (a + b + c [illegible]
16. $(a + b + c)^3 - (b + c)^3 \mp (c + a)^3 + a^3 + b^3 + c^3 - (a+b)^3$
17. $a^2(b - c)^3 + b^2(c - a)^3 + c^2(a - b)^3$. (a − b)(b − c)(c − a) [illegible]
18. ✓ Show that $(a + b + c)^4 - (b + c)^4 - (c + a)^4 - (a + b)^4 + a^4 + b^4 + c^4 = 12abc(a + b + c)$.
19. Show that $(ab + bc + ca)^2 - (a^2b^2 + b^2c^2 + c^2a^2) = 2abc(a + b + c)$.

(*b*)

✓ 1. Show that $(a+b+c+d)^4 + (a+b-c-d)^4 + (a+c-b-d)^4$
$+ (a+d-b-c)^4 - (a+b+c-d)^4 - (a+b-c+d)^4$
$- (a-b+c+d)^4 - (-a+b+c+d)^4 = 192abcd.$

✓ 2. Show that $8(a+b+c)^3 - (a+b)^3 - (b+c)^3 - (c+a)^3$
$= 3(2a+b+c)(a+2b+c)(a+b+2c).$

✓ 3. Show that $2(a+b+c)^3 - (a+b)^3 - (b+c)^3 - (c+a)^3 + 3abc$
$= 3(a+b+c)(ab+bc+ca).$

4. Factor $(x+y+z)^5 - (x^5+y^5+z^5).$

$5(x+y)(y+z)(z+x)($[illegible]$z^2 + xy +$ [illegible] $+ zx)$

5. Factor $a^3 + b^3 - c^3 + 3abc$; $-a^3 + b^3 + c^3 + 3abc$; and
$-a^3 - b^3 + c^3 - 3abc.$ (1) [illegible]

(2) [illegible] (3) $(-a$ [illegible]

6. What value of m will make $x^4 + 13x^3 + 26x^2 + 52x + 7m$ vanish when $x + 11 = 0$? $12\frac{4}{7}$

7. What value of m will make $2x^4 - 10x^2 + 4mx + 6$ vanish when $x^2 - 3x + 3 = 0$? 3

8. $x^3 + ax^2 + bx + c$ vanishes for $x = 3, -2$, and 5; determine a, b, c. $b = -1$, $c = 30$ $a = -6$

9. $x^3 + bx^2 + cx + a$ vanishes for $x = 4, -1$, and 3; determine a, b, c. $b = -6$ $c = 5$ $a = 12$

✓ 10. Show that $(x^2 - 2x + 1)^8 - x^8 + (x^2 - x + 1)^8$ is divisible by $x^3 - 2x^2 + 2x - 1$.

11. Divide $a(b^2 - c^2) + b(c^2 - a^2) + c(a^2 - b^2)$ by
$b(c-a)^3 + c(a-b)^3 + a(b-c)^3$.

12. Divide $(b+c)^3 + (c+a)^3 + (a+b)^3 - 3(a+b)(b+c)(c+a)$ by $a^3 + b^3 + c^3 - 3abc$.

13. Find the factors of $a(b-c)(b+c-a)^4$ + anal. + anal.

✓ 14. Show that $(a-b)(1+ca)(1+cb)$ + anal. + anal.
$= (a-b)(b-c)(c-a)$.

15. Find all the factors of $a(b-c)(1+ab)(1+ac)$ + anal. + anal.

16. Find all the factors of $(b-c)(1+a^2b)(1+a^2c)$ + the two similar terms.

CHAPTER X.

HIGHEST COMMON FACTOR.—LEAST COMMON MULTIPLE.

113. The Highest Common Factor (H. C. F.) of two or more expressions is that expression which is formed by multiplying the greatest common measure (if there be any) of their numerical coefficients into the factor of *highest dimensions* which will exactly divide each of the expressions. Thus,

The H. C. F. of $15x^2y^3z$ and $20x^3y^2z^2$ is the product of

5, the G. C. M. of 15 and 20,
x^2, the highest common factor in x,
y^2, " " " y,
and z, " " " z,

and is therefore $5x^2y^2z$.

114. To find the H. C. F. of two or more **Monomials,** therefore, we have only to *find and multiply together the G. C. M. of the numerical coefficients and the highest common factor of the literal parts.*

EXERCISE LI.

Find the H. C. F. of :—

1. $3x^2y^3$, xy^2 ; $9a^2b^3$, $27a^3b^3c$; $-10a^2b^3c$, $50a^3b^3c^3$; $72b$, $84bc$.
2. $23ab^2c$, $69a^2b^2$; $-5a$, $20a^2b$; $18a^2b^2c^3$, $36a^2bc^3$; $30x^4y^5$, $90x^3y^3z^3$.
3. $42a^2bc^3$, $77a^3b^3c^2$; $12x^{n+1}y^3$, $4x^2y$; $15x^ny^{n-1}z^{n-3}$, $10x^ny^{n-2}z^{n-4}$.
4. $44m^3n^5c^6$, $55m^3n^2c^4$, and $66m^4nc^2$; $60x^3y^2z^6$, $105x^4y^3z^2$, $165x^5y^5$.
5. $72x^2y^3$, $84x^3yz$, and $1728xyz^3$; $39a^4b^3c$, $195a^3b^3c^4$, $104a^3n^3b^2c$.
6. $8a^mb^n$, $24a^{2m}b^{2n}$, and $144a^mb^n$; $12a^{p-1}b^p$, $14a^pb^{p-1}c$.
7. $66a^4b^2c^4$, $44a^6b^4c^5$, and $24a^3b^3c^3$; $24a^3b^3c^5$, $24a^7b^4c^3$, $3a^7b^6c^5$.

115. In like manner when **Polynomials** can be resolved into factors, *the product of the common factors* (if any) *will be the H. C. F. of the polynomials.*

Ex. 1°. Find the H. C. F. of $8a^2x^4 - 24a^2x^3 + 16a^2x$ and $12ax^2y - 12axy - 24ay$.

The first expression

$$= 8a^2x^2 (x^2 - 3x + 2) = 8a^2x^2 (x - 1)(x - 2).$$

The second expression

$$= 12ay (x^2 - x - 2) = 12ay (x + 1)(x - 2).$$

$\therefore$ H. C. F. $= 4a (x - 2)$.

Ex. 2°.—H. C. F. of $4x^4 - 24x^3 - 108x^2$ and $8x^4 - 44x^3 - 252x^2$.

1st expression $= 4x^2 (x^2 - 6x - 27) \quad = 4x^2 (x + 3)(x - 9)$.

2nd " $= 4x^2 (2x^2 - 11x - 63) = 4x^2 (2x + 7)(x - 9)$.

$\therefore$ H. C. F. $= 4x^2 (x - 9)$.

Ex. 3°. H. C. F. of $3x^2 - 6x + 3$, $6x^2 + 6x - 12$ and $12x^3 - 12$.

1st expression $= 3 (x^2 - 2x + 1) = 3 (x - 1)(x - 1)$.

2nd " $= 6 (x^2 + x - 2) \quad = 6 (x + 2)(x - 1)$.

3rd " $= 12 (x^3 - 1) \quad = 12 (x - 1)(x^2 + x + 1)$.

$\therefore$ H. C. F. $= 3 (x - 1)$.

Ex. 4°. Find H. C. F. of $2x^2 + xy - 3y^2$, $3x^2 - 4xy + y^2$ and $2x^2 - xy - y^2$.

1st expression $= (2x + 3y)(x - y)$.
2nd " $= (3x - y)(x - y)$.
3rd " $= (2x + y)(x - y)$.

$\therefore$ H. C. F. $= x - y$.

Ex. 5°. Find H. C. F. of $xy - xz + by - bz$ and $ay - az - y^2 + yz$.

1st expression $= x (y - z) + b (y - z) \quad = (x + b)(y - z)$.

2nd " $= y (a - y) - z (a - y) = (a - y)(y - z)$.

$\therefore$ H. C. F. $= y - z$.

Ex. 6°. Find H. C. F. of $x^3 + 11x^2 - 23x + 39$ and $x^2 + 9x - 52$.

2nd expression $= (x + 13)(x - 4)$.

$x - 4$ is not an exact divisor of the 1st expression, since 4 is not a factor of 39. If, then, there is a common factor, it must be $x + 13$; this is a factor of the expression, for it becomes zero when $x + 13 = 0$.

$\therefore$ H. C. F. $= x + 13$.

Ex. 7°. Find H. C. F. of $2y^3 - (4a - 3c)y^2 + 6(b - ac)y + 9bc$ and $2y^3 + (2a + 3c)y^2 + (3ac - 4b)y - 6bc$.

In the 1st expression only one power of c occurs; factor by Art. 107.

$$3cy^2 - 6acy + 9bc = 3c(y^2 - 2ay + 3b)$$
$$2y^3 - 4ay^2 + 6by = 2y(y^2 - 2ay + 3b)$$

$\therefore$ the 1st expression $= (3c + 2y)(y^2 - 2ay + 3b)$.
Similarly, the 2nd " $= (3c + 2y)(y^2 + ay - 2b)$.

$\therefore$ H. C. F. $= 3c + 2y$.

EXERCISE LII.

(*a*)

Find the H. C. F. of :—

1. $ax(a - x)$, $a^2x(ax + b)$; $3(x + 2)$, $x^2 + 5x + 6$; $4(x - y)^3$, $6(x - y)^2$. ax, $x + 2$, $2($
2. $6(x-a)$, $4(a^2-x^2)$; $a^2b(x-a)(x-b)$, $b^2a(x-a)(x^2-b^2)$.
   $2(x - a)$ $ab(x - a)(x - b)$
3. $x^2 + 7x + 12$, $x^2 + 4x + 3$; $x^2 + 20x + 99$, $x^2 + 10x + 9$.
   $x + 3$ $x + 9$
4. $x^2 - 7x + 10$, $x^2 - 5x + 6$; $x^2 + 9x + 14$, $x^2 + 3x + 2$.
   $x - 2$ $x + 2$
5. $(a + b)^2$, a^2-b^2; x^2+4x+3, x^3+1; x^2-x-6, $x^2+3x-18$.
   $a + b$ $x + 1$ $x - 3$
6. $x^2 + 9x + 20$, $x^2 + 11x + 30$; $x^2 - x - 20$, $x^2 + x - 12$; $(x^2 - 1)^2$, $(x + 1)^2$. $x - 5$; $x + 4$; $(x + 1)^2$
7. $x^2 - 17x - 60$, $x^2 + 23x + 60$; $x^2 - 22xy + 121y^2$, $x^2 - 121y^2$.
   $x - 3$ $x - 11y$
8. $6(x^2 + 9xy + 18y^2)$, $9(x^2 + 3xy - 18y^2)$; $(x + y)^2$, $(x^2 - y^2)^2$; $x^4 - 16$, $x^4 + 4x^2$. $3(x + 6y)$; $x + y$; $x^2 + 4$
9. $x^2 + 5x + 6$, $x^2 + 6x + 8$, $x^2 + 7x + 10$; $x + 2$
   $x^2 + (a + b)x + ab$, $x^2 + (a + c)x + ac$. $x + a$
10. $x^2 - (a - c)x - ac$, $x^2 - (a + c)x + ac$; $x - a$
    $x^2 - y^2$, $x^2 + (4 - y)x - 4y$. $x - y$

11. $x^2 + 3xz,\ x^2 + (2y + 3z)\,x + 6yz$; x + 3
$a^2 + (b + 3)\,a + 3b,\ a^2 + 5a + 6.$ a + 3

12. $a^2 + (b + 3)\,a - (b + 4),\ a^2 - (3 - c)\,a + 2 - c$; a − 1
$x^2 - (a + 3)x - (a + 4),\ x^2 - x\,(a + 4).$ x − 4 − a

13. $a^2 - b^2 + 2bc - c^2,\ a^2 + 2ab - c^2 + b^2$; a
$(a + b + c)^2,\ a^2 - b^2 - c^3 - 2bc.$

14. $(x+y)^2 - 2ab - a^2 - b^2,\ -(y+b)^2 + x^2 + a^2 + 2ax : x^3 + a^3,\ x^2 - a^2.$

15. $x^3 - a^3,\ x^2 + 5ax - 6a^2$; $24\,(x^2 - 9y^2),\ 16\,(x - 3y)^2$;
$x^3 + a^3,\ (x + a)^3.$ x − a, 4(x − 3y),

16. $x^3 + a^2,\ x^2 - a^2,\ x^2 + ax$; $x^3 - 125,\ x^2 - 25,\ x^2 - 10x + 25$
$(1 - x^2)^3,\ (1 - x)^2.$ x + a, x − 5; (1 − x)²,

17. $(x^2 + xy + y^2)^2,\ x^4 + x^2y^2 + y^4$;
$x^6 + a^6,\ (x^2 + ax + a^2)\,(x^2 + a^2)$; $x^4 - y^4,\ (x^2 - y^2)^2.$

18. $15\,(x^6 - y^6),\ 6\,(x + y)\,(x^5 - y^5)$; $12\,(a^4 - b^4),\ 27\,(a^2 + b^2)\,(a + b)^2.$

19. $15\,(p^6 - q^6),\ 25\,(p^2 - q^2)^2$; $x^3 + x^2y + xy^2 + y^3,\ (x^2 - y^2)^2.$

(***b***)

20. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5,\ x^5 - 2x^3y^2 - 2x^2y^3 + y^5.$
x + y

21. $3x^2 - 5x - 2,\ 6x^2 + 5x + 1$; $10x^2 + 13x - 3,\ 5x^2 - 11x + 2.$

22. $10a^2 + 29a + 10,\ 4a^2 + 16a + 15$;
$7a^2 + 33ab - 10b^2,\ 7a^2 + 32ab - 15b^2.$

23. $x^2 + 6x + 9,\ x^3 + 9x\,(x + 3) + 27$; $x^2 - 2x + 1,\ x^3 - 3x\,(x - 1) - 1.$

24. $a^3 - b^3,\ a^3 + 2a^2b + 2ab^2 + b^3$; $a^3 + 2a^2b - ab^2 - 2b^3,\ a^3 + b^2.$

25. $2x^2 + 9x + 4,\ 2x^2 + 11x + 5,\ 2x^2 - 3x - 2$:
$x^5 - xy^2,\ x^3 + x^2y + xy + y^2.$ 2x + 1,

26. $3x^4 + 8x^3 + 4x^2,\ 3x^5 + 11x^4 + 6x^3,\ 3x^4 - 16x^3 - 12x^2.$

27. $x^4 + 4y^4,\ (x^2 + 2y^2 - 2xy)^2$; $x^4 - 7x^2 + 1,\ (x^2 + 3x)^2 - 1.$

28. $6x^3 - 7ax^2 - 20a^2x,\ 3x^2 + ax - 4a^2$;
$21x^3 - 17ax + 6x^2,\ 9x^2 + 6ax - 8a^2.$

29. $(x^2 - xy + y^2)^2 - (x^2 + xy + y^2)^2,\ 4\,(x^2 - y^2)^2$;
$1 - \frac{1}{a^4},\ 1 + \frac{1}{a^2} + \frac{2}{a}.$

30. $2 + a^4 + \frac{1}{a^4},\ a^4 - \frac{1}{a^4}$;
$x^3 + 3x^2 - 13x - 15,\ x^2 - 6x - 55.$

31. $2ax^2 - (a-2)x - 1,\ 2bx^2 - (b+2)x + 1$;
$3x^2 - 22x + 32,\ x^3 + 32x - 28.$

32. $4a^2 - 9b^2 + 6bc - c^2,\ 4a^2 + 9b^2 - 2c^2 + 12ab + 2ac + 3bc.$

33. $(2x + 3y - z)^3 + (3x + 7y + z)^3,\ 25(x^2 - 4y^2)^2.$

34. $(m^2 - 3m + 2)x^2 + (2m^2 - 4m + 1)x + m(m-1),$
$m(m-1)x^2 + (2m^2 - 1)x + m(m+1).$

35. $2ap^3 + (3a - 2b)p^2q + (a - 3b)pq^2 - bq^3$ and
$3ap^3 - (a + 3b)p^2q + (2a + b)pq^2 - 2bq^3.$

36. $2ax^3 + (4a^2 - 1)bx^2 - (2ab^2 + 3c)x - 6abc$ and
$ax^3 - (3 - 2a)bx^2 + (2c - 6ab^2)x + 4abc.$

37. $a^2x^3 + a^5 - 2abx^3 + b^2x^3 + a^3b^2 - 2a^4b,$
$2a^2x^4 - 5a^4x^2 + 3a^6 - 2b^2x^4 + 5a^2b^2x^2 - 3a^4b^2.$

38. $6ax^2 - 6ab^2 - 54a - 21x^2 + 21b^2 + 189,$
$6ab^3 - 12a - 21b^3 + 42.$

39. $6a^3x^2 + 4ax^3 - 10axy - 3a^2xy - 2x^2y + 5y^2,$
$10a^2x^2 - 2ax^3y - 6axy^2 - 5axy + x^2y^2 + 5y^2$, and
$-4ax^2 - 6a^2bxy + 2ab^2x^2y - 2abx + 2xy + 3aby^2 - b^2xy^2 + by.$

× 40. $x^3 - 3x^2 + 3x - 1,\ x^3 - x^2 - x + 1,\ x^4 - 2x^3 + 2x - 1$, and
$x^4 - 2x^3 + 2x^2 - 2x + 1.$ $(x-1)^2$

116. When two expressions cannot readily be resolved into factors, their H. C. F. may be found by a method which depends on the following simple principles :—

1°. *If an expression has a factor, any multiple of that expression will have that factor ;* thus, ax contains the factor a, and it is self-evident that any multiple of ax, as $3ax$, $5ax$, max, contains the same factor a.

2°. *If two expressions have a common factor, then the sum or the difference of any multiples of these expressions will have that common factor ;* thus, ax and ay have the common factor a, and it is evident that $max \pm nay$ have the same common factor a, for the expression $= a\,(mx \pm ny)$. Thus if A and B, any two expressions, have a common factor, that factor is also contained in $mA \pm nB$.

The method referred to will be understood from the following examples. It is generally best to work with detached coefficients; we give examples in both ways.

Ex. 1. Find the H. C. F. of the expressions designated A and B.

$$\begin{array}{r|l}
A & x^3 + 3x^2 + 4x + 12 \\
B & x^3 + 4x^2 + 4x + 3 \\ \hline
B - A = C & x^2 - 9 \\ \hline
4 \times B = & 4x^3 + 16x^2 + 16x + 12 \\
A = & x^3 + 3x^2 + 4x + 12 \\ \hline
(4B - A) \div x = D & 3x^2 + 13x + 12 \\
3 \times C = & 3x^2 - 27 \\ \hline
(D - 3C) \div 13 = E & x + 3 \\ \hline
3 \times D = & 9x^2 + 39x + 36 \\
4 \times C = & 4x^2 - 36 \\ \hline
(3D - 4C) \div 13x = F & x + 3 \\ \hline
\end{array}$$

$\therefore$ E (or F), *i. e.* $x + 3$ is the H. C. F.

1°. We operate on A and B to *cancel* the first terms and the last; this gives us C and D.

2°. We operate on C and D in the same way; this gives us E and F, which are identical.

3°. We introduce or strike out any factor which is manifestly no part of the required H. C. F.

We are not *obliged* to operate on the first and last terms, but in general it is convenient to do so.

From the principles stated at the beginning of this article, it is evident that the H. C. F. of A and B must be found in C and D, and $\therefore$ also in E and F.

If C and D had been identical, one of them would have been the required H. C. F.

Also, we might *factor* C and D, and thus determine the *common* factor; the object of working the example, however, is to exhibit the METHOD.

Ex. 2. Find the H. C. F. of the following expressions, A and B.

$$\begin{array}{r|ll}
\text{A} & 6x^3 + x^2 - 4x + 21 & \\
\text{B} & 3x^3 - 13x^2 + 23x - 21 & \\
\hline
(\text{A} + \text{B}) \div 3x = \text{C} & 3x^2 - 4x - 7 & = (3x - 7)(x+1) \\
\hline
2\text{B} = & 6x^3 - 26x^2 + 46x - 42 & \\
\text{A} = & 6x^3 + x^2 - 4x + 21 & \\
\hline
(\text{A} - 2\text{B}) \div 9 = \text{D} & 3x^2 - 10x + 7 & = (3x - 7)(x-1) \\
\hline
(\text{C} + \text{D}) \div 2x = \text{E} & 3x - 7 & \\
(\text{C} - \text{D}) \div 2 = \text{F} & 3x - 7 & \\
\hline
\end{array}$$

$\therefore\ 3x - 7$ is the H. C. F.

Ex. 3. Find the H. C. F. of the expressions marked A and B.

$$\begin{array}{r|ll}
\text{A} & 4x^3 - 6x^2 - 4x + 3 & \\
\text{B} & 6x^3 + x^2 - 1 & \\
\hline
3 \times \text{A} = & 12x^3 - 18x^2 - 12x + 9 & \\
2 \times \text{B} = & 12x^3 + 2x^2 - 2 & \\
\hline
2\,\text{B} - 3\text{A} = \text{C} & 20x^2 + 12x - 11 & =(2x-1)(10x+11) \\
3 \times \text{B} = & 18x^3 + 3x^2 - 3 & \\
\hline
\text{A} = & 4x^3 - 6x^2 - 4x + 3 & \\
\hline
(\text{A} + 3\text{B}) \div x = \text{D} & 22x^2 - 3x - 4 & =(2x-1)(11x+4) \\
\hline
4 \times \text{C} = & 80x^2 + 48x - 44 & \\
11 \times \text{D} = & 242x^2 + 33x - 44 & \\
\hline
(11\text{D} - 4\,\text{C}) \div 81x = \text{E} & 2x - 1 & \\
\hline
11 \times \text{C} = & 220x^2 + 132x - 121 & \\
10 \times \text{D} = & 220x^2 - 30x - 40 & \\
\hline
(11\text{C} - 10\,\text{D}) \div 81 = \text{F} & 2x - 1 & \\
\hline
\end{array}$$

$\therefore\ 2x - 1$ is the H. C. F.

1°. We might have obtained the required factor at once from C, since it is plain that only *one* of its factors, $2x - 1$, can divide A and B.

2°. Or, we might have stopped with D, and obtained the required factor by factoring C and D.

Ex. 4. Find the H. C. F. of $3x^2+x-2$ and $3x^2+4x-4$.

$$\begin{array}{r|rrr}
A & 3 & +1 & -2 \\
B & 3 & +4 & -4 \\
\hline
B-A=C & & 3 & -2 \\
\hline
2\cdot A= & 6 & +2 & -4 \\
B= & 3 & +4 & -4 \\
\hline
(2A-B)\div x=D & 3 & -2 & \\
\hline
\end{array}$$

We use detached coefficients and designate the given expressions as A and B, and the resulting expressions as C, D, etc.

Every *common* factor of A and B must appear in B — A *i. e.* in C. So every common factor must appear in 2 A — B *i. e.* in D: but C and D are identical, and can therefore have in them nothing different from the common factor.

∴ C or D, that is $3x-2$ is the H. C. F.

Ex. 5. Find the H. C. F. of $4x^5+11x^4+81$ and $4x^5-22x^2-18$.

$$\begin{array}{r|rrrrrr}
A & 4 & +11 & +0 & +0 & +0 & +81 \\
B & 4 & +0 & +0 & -22 & +0 & -18 \\
\hline
\frac{1}{11}(A-B)=C & & 1 & +0 & +2 & +0 & +9 \\
\hline
2\cdot A= & 8 & +22 & +0 & +0 & +0 & +162 \\
9\cdot B= & 36 & +0 & +0 & -198 & +0 & -162 \\
\hline
2A-9B= & 44 & +22 & +0 & -198 & +0 & +0 \\
\frac{1}{22x^2}(2A-9B)=D & 2 & +1 & +0 & -9 & +0 & \\
\hline
2\cdot C= & 2 & +0 & +4 & +0 & +18 & \\
D-2C=E & 0 & +1 & -4 & -9 & -18 & \\
\hline
D= & 2 & +1 & +0 & -9 & & \\
\hline
2\cdot E= & 2 & -8 & -18 & -36 & & \\
\frac{1}{9}(D-2E)=F & & 1 & +2 & +3 & & \\
\hline
2\cdot D= & 4 & +2 & +0 & -18 & & \\
E= & 1 & -4 & -9 & -18 & & \\
\hline
\frac{1}{3x}(2D-E)=G & 1 & +2 & +3 & & & \\
\hline
\end{array}$$

∴ $x^2+2x+3=$ H. C. F.

As before, work on A and B to get C and D; then work on C and D to get E and F, etc. It is plain that none of the factors struck out or introduced, form any part of the common factor sought for.

Ex. 6. Find the H. C. F. of $x^4 - 4x^3 + 6x^2 - 4x + 1$ and $x^4 - 2x^2 + 1$.

$$\begin{array}{r|c}
\text{A} & 1 - 4 + 6 - 4 + 1 \\
\text{B} & 1 + 0 - 2 + 0 + 1 \\
\hline
\frac{1}{4}(\text{B} - \text{A}) = \text{C} & 1 - 2 + 1 \\
\hline
\end{array}$$

$\therefore x^2 - 2x + 1$ is the H. C. F.

Ex. 7. Find the H. C. F. of $x^4 - ax^3 - a^2x^2 - a^3x - 2a^4$ and $3x^3 - 7ax^2 + 3a^2x - 2a^3$.

$$\begin{array}{r|l}
\text{B} & 0 + 3 - 7 + 3 - 2 \\
\text{A} & 1 - 1 - 1 - 1 - 2 \\
\hline
x\text{B} = & 3 - 7 + 3 - 2 + 0 \\
3\text{A} = & 3 - 3 - 3 - 3 - 6 \\
\hline
(3\text{A} - x\text{B}) \div a = \text{C} & \quad\;\; 4 - 6 - 1 - 6 \\
\hline
\text{A} = & 1 - 1 - 1 - 1 - 2 \\
\text{B} = & 0 + 3 - 7 + 3 - 2 \\
\hline
(\text{A} - a\text{B}) \div x = \text{D} & 1 - 4 + 6 - 4 \\
\hline
4\text{D} = & 4 - 16 + 24 - 16 \\
\text{C} = & 4 - 6 - 1 - 6 \\
\hline
\frac{1}{5}(\text{C} - 4\text{D}) = \text{E} & \quad\;\; 2 - 5 + 2 \quad \left\{\begin{array}{l} = x^2 - 5ax + 2a^2 \\ = (x - 2a)(2x - a) \end{array}\right. \\
\hline
3\text{D} = & 3 - 12 + 18 - 12 \\
2\text{C} = & 8 - 12 - 2 - 12 \\
\hline
\frac{1}{5}(2\text{C} - 3\text{D}) = \text{F} & 1 + 0 - 4 = x^2 - 4a^2 = (x + 2a)(x - 2a) \\
\hline
\end{array}$$

$\therefore$ Comparing E and F, we see that $x - 2a$ is the H. C. F.

117. We add one example solved by the common rule for finding the H. C. F. of two expressions A and B. Divide A by B (if A is not of lower dimensions than B); make the remainder (R_1 say) a new *divisor* with B for *dividend*; then make the new remainder (R_2 say) a *divisor* with R_1 for *dividend*. Proceed in this way till there is no remainder; the *last divisor* is the H. C. F.

required. This method depends on the same principles as the other.

N. B.—In both methods, a factor may be removed, and, to avoid fractions, a factor may be introduced, at any stage of the operation; also, all the signs of any expression may be changed.

Ex. 8. Find the H. C. F. of $4x^5+11x^4+81$ and $2x^5-11x^2-9$. *The right-hand quotients are from right-hand dividends; the left-hand quotients from left-hand dividends.*

$$
\begin{array}{r|r||r|l}
\text{QUOTIENTS.} & & & \text{QUOTIENTS.} \\
2 & 4x^5+11x^4+81 & 2x^5-11x^2-9 & 2x \\
 & 4x^5-22x^2-18 & 2x^5+4x^3+18x & \\
\cline{2-4}
 & 11)\,11x^4-22x^2+99 & -4x^3-11x^2-18x-9 & -4x \\
\cline{2-2}
 & x^4+2x^2+9 & -4x^3-8x^2-12x & \\
\cline{3-3}
 & 4 & -3x^2-6x-9 & -3 \\
\cline{1-2}
-x & 4x^4+8x^2+36 & -3x^2-6x-9 & \\
\cline{3-3}
 & 4x^4+11x^3+18x^2+9x & & \\
\cline{2-2}
 & -11x^3-10x^2-9x+36 & & \\
 & 4 & & \\
\cline{2-2}
+11 & -44x^3-40x^2-36x+144 & & \\
 & -44x^3-121x^2-198x-99 & & \\
\cline{2-2}
 & 81)\,81x^2+162x+243 & & \\
\cline{2-2}
 & x^2+2x+3 & &
\end{array}
$$

$\therefore\ x^2+2x+3$ is the H. C. F.

118. If the H. C. F. of more than two expressions is to be found, find the H. C. F. of two of them, then the H. C. F. of that and another of the expressions, and so on.

NOTE.—The H. C. F. of an algebraic expression is not necessarily the *greatest common measure* of the *value* of the expression when numerical values are assigned to the letters; *e. g.*, The algebraic H. C. F. of x^2-1 and $(x+1)^2$ is $x+1$. Suppose $x=11$, then the value of this H. C. F. is **12**; but the *greatest common measure* of the *arithmetical values* of the expressions, (11^2-1) and $(11+1)^2$ is **24**. It is better to use the term H. C. F. *in algebra*, as it is more significant than *greatest common measure*.

EXERCISE LIII.

(*a*)

Find the H. C. F. of the following quantities :—

1. $2x^3 + 6x^2 + 6x + 2,\ 6x^3 + 6x^2 - 6x - 6.$ $2(x+1)^2$

2. $x^2 - 7x + 10,\ 4x^3 - 25x^2 + 20x + 25.$ $x-5$

3. $6x^3 - 13x^2 + 3x + 2,\ 10x^3 - 23x^2 + 7x + 4.$ $2x^2-3x-1$

4. $7a^3 - 46a^2 - 22a + 7,\ 49a^3 + 49a^2 + 5a - 4.$ $7a^2+3a$ [illegible]

5. $7y^3 + 52y^2 - 46y + 8,\ 3y^3 + 17y^2 - 62y + 14.$ y^2+8y-2

6. $3y^3 - 2y^2 - 36y - 35,\ 2y^3 - 17y^2 + 23y + 55.$ y^2-3y-5

7. $3x^4 - 10x^3 + 5x^2 - 18,\ 3x^4 - 12x^3 + 7x^2 + 10x - 12.$ x^2-2x-3

8. $5x^3 + 2x^2 - 15x - 6,\ -7x^3 + 4x^2 + 21x - 12.$ x^2-3

← 9. $20x^4 + x^2 - 1,\ 25x^4 + 5x^3 - x \overset{-}{+} 1.$ $5x^2-1$

10. $6x^4 - x^3y - 3x^2y^2 + 3xy^3 - y^4,\ 9x^4 - 3x^3y - 2x^2y^2 + 3xy^3 - y^4.$ $[3x^2-2xy+y^2$

11. $x^4 - 2x^3 + x^2 - 8x + 8,\ 4x^3 - 12x^2 + 9x - 1.$ $x-1$

12. $x^4 + 6x^3 + 11x^2 + 4x - 4,\ x^4 + 2x^3 - 5x^2 - 12x - 4.$ x^2+4x+4

13. $12x^5 - 12x^3y^2 + 12x^2y^3 - 3xy^4,\ 12x^5 + 8x^4y - 18x^3y^2 - 6x^2y^3 + 4xy^4.$ $x($[illegible]x^2+2xy-[illegible]$)$

14. $x^3 - 9x^2 + 26x - 24,\ x^3 - 10x^2 + 31x - 30,\ x^3 - 11x^2 + 38x - 40.$ $x-2$

15. $x^4 - 10x^2 + 9,\ x^4 + 10x^3 + 20x^2 - 10x - 21,$ and
$x^4 + 4x^3 - 22x^2 - 4x + 21.$ x^2-1

(*b*)

Find the H. C. F. of the following expressions :—

1. $x^3 + 2x^2 - 13x + 10,\ 3x^3 + 3x^2 - 30x + 24.$

2. $4x^3 - 20x^2 - 396x + 160,\ 2x^3 - 12x^2 - 172x + 70.$

3. $a^3 + 2a^2 - 8a - 16,\ a^3 + 3a^2 - 8a - 24.$

4. $4x^3 - 32x^2 + 85x - 75,\ 3x^3 - 15x^2 + 15x + 9.$ $x-3$

5. $48x^4 + 8x^3 + 31x^2 + 15x,\ 24x^4 + 22x^3 + 17x^2 + 5x.$

6. $6x^5 - 4x^4 - 11x^3 - 3x^2 - 3x - 1,\ 4x^4 + 2x^3 - 18x^2 + 3x - 5.$

7. $6x^3 + 13x^2 + 15x - 25,\ 10x^3 + 20x^2 + 20x - 50.$ $x^2 + 3x + 5$

8. $3x^5 - x^4 - 3x + 1,\ 3x^4 + x^3 + x^2 + x - 2.$ $x^3 + x^2 + x + 1$

9. $a^7 - 3a^4 - 4a^3 - 3a^2 - a,\ a^6 - a^4 - 2a^3 - 3a^2 - 2a - 1.$
$a^3 - a^2 - a - 1$

10. $2y^6 + 2y^5 - 3y^4 - 5y^3 - 14y^2 - 7y$ and
$2y^5 + 2y^4 - 5y^3 - 5y^2 - 7y - 7.$ $2y^2 - 7$

11. $2x^6 - x^5 - 5x^3 - 5x^2 - x,\ 2x^5 - x^4 - 2x^3 - 2x^2 - 4x - 1.$
$2x^3 - 3x - 1$

12. $x^6 + 4x^5 - 3x^4 - 16x^3 + 11x^2 + 12x - 9$ and
$6x^5 + 20x^4 - 12x^3 - 48x^2 + 22x + 12.$ $x^3 + x^2 - 5x + 3$

13. $72x^3 - 12ax^2 + 72a^2x - 420a^3,\ 18x^3 + 42ax^2 - 282a^2x + 270a^3.$
$6(3x - 5a)$

14. $4x^5 + 14x^4 + 20x^3 + 70x^2,\ 8x^7 + 28x^6 - 8x^5 - 12x^4 + 56x^3.$
$2x^3(2x+9)$

15. $x^6 - 7x^5 - 3x^4 - 5x^3 + 42x^2 - 34x - 21,$
$x^5 - 11x^4 + 25x^3 + 19x^2 - 49x - 21.$ $x^2 - 7x - 3$

119. The Least Common Multiple (L. C. M.) of two or more expressions is the expression of *lowest dimensions* which is *exactly divisible* by each of the expressions.

It is clear that if an expression is exactly divisible by, or exactly "contains," a certain divisor, it must be divisible by, or contain, *all the factors* of that divisor; on this axiom depends the rule for finding the L. C. M. in both Arithmetic and Algebra.

In Arithmetic the L. C. M. of two or more numbers must contain the *different factors* of the numbers, and *each in its highest power.* Thus to find the L. C. M. of 440 and 572; their factors are $2^3 \cdot 5 \cdot 11$ and $2^2 \cdot 11 \cdot 13$: The *different* factors are 2, 5, 11, 13, and the L. C. M. must contain *all* these factors *each* in its *highest* power, *i. e.* it must contain 2^3, 5, 11, and 13.

So the L. C. M. of $3a^4bc^3$, and $5a^3b^2c^3d$, must contain 3, 5, a, b, c, and d, and each in the highest power in which it occurs in either of the quantities; it is, therefore, $15a^4b^2c^3d$. So in $x - 1$, $x + 1$, $x^2 - 1$, $(x + 1)^2$, $(x - 1)^3$, the *different* factors are, $x - 1$, and $x + 1$; the *second* power of $x + 1$ occurs, and the *third* power of $x - 1$ $\therefore$ H. C. F. is $(x + 1)^2(x - 1)^3$.

120. Hence, **to find the L. C. M. of two expressions,** we have only to *multiply together the different factors each in the highest power in which it occurs.*

EXERCISE LIV.

(a)

Write down, in factors, the Least Common Multiple of :—

1. $4a^2$, $2a^3$; $4x^2$, $6yz$; abx, cx^2y ; $6a^2b$, $12ab^2$; $4xyz^2$, $7x^2yz$.

2. $42a^2b^2$, $14a^3b$; x^4, $4x^2y^2$; $10x^4y^3$, $5xy^4$; $7a^2b^3c^2$, $3abc^3$.

3. x^3y^2, x^2y^3 ; $2ab$, $3a^2b^3$; $7a^2$, $3b^2$, $6ab$; ab, bc, ca.

4. $2xy^2$, $3x^2y$, $4x^2y^2$; $7x^4y$, $8xy^4$, x^4y^4 ; $2xy$, $3x^2y^2$, $4x^3y^3$; ax^3, $6xy$.

5. $2abc$, $3bca$, $4cab$; $3xy$, $4x^2y$, $5xy^2$; $3p^2q$, $6pq^2$, $2p^3$; ax^ny, by^n.

6. abc^3, $a^2b^2c^4$; $15a^3x^2y$, $5ax^3y^4$; m^2n^2p, $3m^3p^4$; p^nq, q^np.

7. $3x$, $3(a-x)$; $3a^2b$, $3ab(a+b)$; a, $(a-b)^2$; abc, $ab(a^2-c^2)$.

8. $2a^2(a+x)$, $4ax$; $3(x+y)$, $7(a+b)$; $a^2b(p+q)$, $ab^2(p-q)$.

9. a, $a+b$, $b+c$; $x+1$, x ; $x-3$, x^2 ; $x-1$, $x+1$; a^3, $(a-b)^3$.

10. $a(x+a)$, $b(x+b)$, ab ; $a(x-a)$, $b(x+a)$; $x+1$, $x-1$, x^2-1.

11. $x-2$, $x-1$; $3(x-2)$, $7(x-2)(x+2)$; x, $x+1$, $x+2$; $(x+y)^2$, $(x+y)^3$.

12. $(x+1)(x+2)$, $(x+2)(x+3)$, $(x+3)(x+4)$; $a+b$, $a-b$, (a^2-b^2).

13. $(3x-2)(2x-5)$, $(2x-5)(x+7)$, $(x+7)x$; $x^2(a-b)$, $xy(a+b)$, a^2-b^2.
$3x-2$ $2x-5$ $(x+7)x$; $yx^2(a$ $b)(a+b)$

14. $x(x-a)$, $x^2(x+a)$, $x^3(x^2-a^2)$; $(x-a)(x-b)$, $(x-b)(x-c)$, $a^3(x-c)$.
$x^3(x-a)(x+a)$; $x^3(x-a)(x-b)(x-c)$

15. $3(x-y)^2$, $6(x-y)^2$; $6(x+y)^5$, $3(x+y)^3$; x^2-a^2, $(x-a)(x+b)$.
$6(x+y)^5$ $x-a$ $+a)(x+b)$

16. $3x+1$, $2x-5$, $9x^2-1$; $a(a+3)^2$, $a^3(a+3)$, a^4 ; $(a^2-b^2)^2$, $(a+b)^3$.
$(3x+1)(3x-1)(2x-5)$, $a^4(a+3)^2$; $(a+b)^3(a-$

17. $(a-x)^2$, $(a+b)(a-x)$, $(a+b)^2$; $(a+b)^3$, a^3+b^3, a^2-ab+b^2.
$(a-x)^2(a+b)^2$ $(a+b)^3(a^2-$ [illegible] $+b^2)$

18. $(a-x)^2$, $b-y$, $(b-y)^2$, $(a-x)(b-y)^3$; $2(a-1)$, $3(a+1)$, $6(a^2-1)^2$.
$(a-x)^2(b-y)^3$; $6(a+$ [illegible]

19. $2x+1$, $2x-1$, $1-4x^2$; $(x+y)^2$, $(x-y)^2$, x^2-y^2; $a^n(a-1)$, $b^n(1+b)$. [illegible] $(x^2-y^2)^2$ [illegible]

20. $2a^n(b-1)$, $3b^n(1-b^2)$; $x+y$, $x-y$, x^2-y^2; $x+1$, $x-1$, x^2-1, x^2+1. $-6a^nb^n(b^2-1)$; x^2-y^2; $x^2-1)$ [illegible]

(b)

Find the L. C. M. of :—

21. $4a^2b-b$, $2a^2+a$; $6x^2-2x$, $9x^2-3x$; x^2+2x, x^2-4.
$(4a^2b-b)$ $6x(3x-1)$ $x(x-2)(x+2)$

22. x^2-1, $(x-1)^2$; x^2-5x+4, x^2-6x+8; x^2-x-6, x^2+x-2.
$(x+1)(x-1)^2$ $(x-1)(x-2)(x-4$ $(x-1)(x+2)(x-3)$

23. $x^2+8x+15$, $x^2+9x+20$; x^2+x-2, x^2-2x+1.
$x+3)($ [illegible] $x+5)$ $(x-1)^2(x+2)$

24. a^2+a-2, a^2-3a+2; $(x-y)^2$, $(x+y)^2$, x^2-y^2, x^2y^2.
$(a-1)(a-2)(a+2)$ $x^2y^2(x^2-y^2)^2$ or $(x-y)^2(x+y)^2x^2y^2$

25. $x^2+2x-120$, x^2-2x-8; $x^2-15x+36$, $x^2-9x-36$.
$(x+2)(x-4)(x-10)(x+12)$ $(x+3)(x-3)(x-12)$

26. $x^2-9x+14$, $x^2-11x+28$; x^2-3x-4, x^2-x-12.
$x-2$ [illegible] $-4)$ $(x+1)(x+3)(x-4)$

27. $x^2+(a+b)x+ab$, $x^2+(a-b)x-ab$; $1+x$, $1-x$, $(x+1)^2$.
$x+a$ [illegible] x^2-b^2 $1-x)(1+x)^2$

28. x^2-a^2, $x^2-(a+b)x+ab$; $(a+b)^2-c^2$, $(a+b+c)^2$.
$(x-a)(x+a)(x-b)$ $a+b-c)(a+b+c)^2$

29. $(x+2)^2$, $(x-2)^2$, $(x^2-4)^2$; $2x^2-7x+3$, $2x^2+5x-3$.
[illegible] $x+3)(x-3)(2x-$

30. $3y^2+7y+2$, y^2-y-6; $6x^2-13x+6$, $6x^2+5x-6$.
$(y+2)(y-3)(3y+1)$ [illegible] $(3x-2)$

31. $6x^2-13xy+6y^2$, $4x^2-9y^2$; $9x^2+3x-2$, $9x^2-3x-6$.
$(2x+3y)(2x-3y)(3x-2y)$ $3x-1)3$ [illegible]

32. $60a^4+5a^3-5a^2$, $60a^2y+32ay+4y$; $40a^3y-2a^2y-2ay$, $16a^2-1$.
$20a^2y(4a-1)(a+1)(3a+1)$ $2ay(4a-1)$ [illegible]

33. $3x^2-x-14$, $3x^2-13x+14$, x^2-4; $x^{2n}+y^n$, $x^{2n}-y^n$.
$(x+2)(x-2)(3x-7)$ [illegible]

34. x^2-5x+6, $x^2-7x+12$, $x^2-9x+20$; $4ax^2y^2+11axy^2-3ay^2$ and $24ax^2-22ax+4a$. $(x-2)$ [illegible]
$2ax^2(4x-1)(x+3)(3x-2)$

35. $x+y$, $x-y$, x^2+y^2, x^4-y^4; $4(x+1)^2$, $6(x^2-1)$, $(x-1)^2$.

36. x^2-a^2, $(x+a)^2$, $(x-a)^2$; $6x$, $(x+1)^3$, $4(x^2-1)$, $(1-x)$.
$(x+a)^2(x-a)^2$ [illegible]

37. $c(a+b)$, $b(a-b)$, $c(a^2-b^2)$; x^2, $(x+1)^2$, $(x^2-1)^2$.
$=(a+c)(a-b)$ $x^2(x+1)^2(x-1)^2$

38. $x+y$, $x-y$, x^3-y^3; $8(1-x)$, $4(1+x)$, $12(1+x^2)$.
$(x+y)(x^3-y^3)$ $24(1-x)(1+x)(1+x^2)$

39. $ax+by$, $ax-by$, $a^2x^2+b^2y^2$; $a+b$, $a-b$, a^2-b^2, a^4+b^4.
$a^4x^4-b^4y^4$ $(a^2-b^2)(a^4+b^4)$

40. $4a^3(a+y)$, $6a^2(a-y)$, $2a^3(a^2-y^2)$; x^2-ax+a^2, x^3+a^3, $x+a$.
$12a^3(a^2-y^2)$ [illegible]

41. $1 - x^3$, $1 + x + x^2$, $1 - x$; $1 - 8x^3$, $1 - 4x + 4x^2$; $x^3 + y^3$, $(x + y)^3$.

$1-x^3$ $(1-2x)^2(1-8x^3)$ $(x+y)^3(x^2-xy+y^2)$

42. $x^2 - x + 1$, $x^2 + x + 1$, $x^2 + 1$; $x^2 - xy + y^2$, $x^2 + xy + y^2$, $x^3 - y^3$.

$(x+1)(x^4+x^2+1)$ $(x-y)(x^4+x^2y^2+y^4)$

43. $x^2 - y^2$, $x^2 - xy + y^2$, $x^2 + xy + y^2$; $(x + 4)^3$, $x^3 + 64$; $x^{2n} - 1$, $x^2 - 1$.

x^6-y^6 $(x+4)^3(x^2-4x+16)$ $x^{2n}-1$

44. $1 + x + x^2$, $1 - x + x^2$, $1 + x^2 + x^4$; $x^2 - y^2$, $x^3 - y^3$, $x^2 - xy + y^2$.

$1+x^2+x^4$ x^6-y^6

45. $(a - b)(a - c)$, $(b - c)(c - a)$, $(c - a)(a - b)$; $(x^2 - y^2)(y^2 - z^2)$, $(y^2 - z^2)(z^2 - x^2)$, $(z^2 - x^2)(x^2 - y^2)$.

$(a-b)(a-c)(b-c)$; $(x^2-y^2)(y^2-z^2)(z^2-x^2)$

121. If two expressions cannot be easily factored, their H. C. F. is to be obtained as in Art. 116. Then the L. C. M. *can be found by dividing* **either** *of the expressions by the H. C. F. and multiplying the quotient by the other.* Thus,

Let a and b have the H. C. F. m, and

Let $$\frac{a}{m} = q, \quad \therefore \quad a = mq;$$

and $$\frac{b}{m} = q', \quad \therefore \quad b = mq'.$$

The L. C. M. of a and b is, of course, that of

$$mq \text{ and } mq'$$

$$= mqq' = \frac{mq}{m} \times mq' = \frac{a}{m} \times b;$$

or $$= \frac{mq'}{m} \times mq = \frac{b}{m} \times a.$$

122. If the L. C. M. of *more than two* expressions is to be found, we find the L. C. M. of *two* of them, then of that L. C. M. and the third expression, and so on.

EXERCISE LV.

Find the L. C. M. of :—

1. $x^3 + 6x^2 + 11x + 6$, $x^3 - 6x^2 - 25x + 150$. $(x+1)(x+2)(x+3)(x-$ [illegible]

2. $x^3 + x^2 + x + 1$, $6x^5 + 5x^4 + 8x^3 + 4x^2 + 2x - 1$. $x+1$

3. $x^3 + 10x^2 + 31x + 30$, $x^3 + 9x^2 + 26x + 24$. $(x+5)$

4. $x^3 + 12x^2 + 47x + 60$, $x^3 + 13x^2 + 56x + 80$.

5. $2x^3 + x^2 - 11x - 12,\ 6x^3 + 9x^2 - 39x - 60.$

6. $4 - 13x + 11x^2 - 2x^3,\ 4x + 3x^2 - 9x^3 + 2x^4.$

7. $2x^4 - 4x^3 - 6x^2 + 16x - 8,\ 3x^4 - 15x^3 + 60x - 48.$ $(x - 2)$

8. $4x^3 + 18x^2 + 13x + 20,\ 4x^4 + 2x^3 - 3x^2 - 4x - 10.$

9. $3x^3 + 20x^2 + 31x + 30,\ 3x^4 + 2x^3 + 4x^2 - x + 6.$

10. $4a^3 + 23a^2 + 24a + 45,$
$28a^4 + 21a^3 + 63a^2 + 36a^3b + 27a^2b + 81ab.$ $a + 5)$

11. $3x^3 - 5x^2 + 5x - 2,\ 2x^3 + x^2 - x + 3,\ 6x^3 - x^2 - 11x + 6.$ $(x^2 - x + 1)$

$6x^5 - 7x^4 - 4x^3 + 16x^2 - ...$

12. $x^3 - 5x^2 + 9x - 9,\ x^3 - x^2 - 9x + 9,\ x^4 - 4x^2 + 12x - 9.$

$(x^2 - 2x + 3)(x^3 - x^2 - 9x + 9) = x^5 - 3x^4 - 4x^3 + 24x^2 - ...$

13. $6x^5 + 13x^4 + 2x^3 + 2x^2 + 1,\ 9x^5 + 5x^3 - x^2 - 1.$ $(2x^3 + 2x^2 + 2x + 1)$

14. $x^3 + 6x^2 + 11x + 6,\ x^3 + 7x^2 + 14x + 8,$
$x^3 + 8x^2 + 19x + 12,\ x^3 + 9x^2 + 26x + 24.$

$(x + 3)(x^3 + 7x^2 + 14x + 8) = x^4 + 10x^3 + 35x^2 + 50x + 24$

15. $3x^4 - 10x^3 + 5x^2 - 18,\ 3x^4 - 12x^3 + 7x^2 + 10x - 12.$ $x + 6$

$(3x^2 - 6x + 11)(3x^4 - 10x^3 + 5x - 1...$

EXERCISE LVI.

(*a*)

−30 1. If $x - 3$ is a measure of $x^2 + 7x + a$, what is the value of a?

2. If $x + 4$ measures $x^2 - x - a$, find a. 20

3. Find values for a and b, in order that $x - 3$ may be a common factor of $x^2 - 7x + a$ and $x^2 + x - b$. $a = b = 12$

4. What value of a will make $x - 7$ a measure of $x^2 - ax + 21$? 10

5. Find the values of a and b, when $x + 7$ is a common measure of $x^2 - ax + 21$ and $x^2 + bx - 56$. $a = -10$ $b = -1$

6. $x + 3$ is a measure of $x^3 - 6x^2 + 11x - c$, find c. -114

7. $x - 4$ is a measure of $x^3 - 7x^2 + cx - x$, find c. 13

8. $x - 2$ is a common measure of $x^3 - x^2 - cx + 40$ and $x^3 - 12x^2 + 44x - a$, find c and a. $c = 22$ $a = 48$

9. If $x - a$ is a measure of $x^2 + bx - a$, to what is b equal? $1 - 2$

10. If $4x^2 + bx - 20$ is a multiple of $2x + 5$, find b. 7

11. If $ax^2 + bx - c$ is a multiple of $x + b$, to what is c equal? $-b^2(1-a)$

✓ 12. If $x + 1$ and $x - 1$ are both measures of $x^3 + px^2 + qx + r$, show that $p = -r$.

✓ 13. If $x^n + py^n + qz^n$ is exactly divisible by $x^2 - (ay + bz)x + abyz$, then $p \div a^n + q \div b^n + 1 = 0$.

✓ 14. If $x + a$ is a common factor of $x^2 + px + q$ and $x^2 + lx + m$, prove that $a = \dfrac{m - q}{l - p}$.

15. $x + a$ is a common factor of $x^2 + px + 1$ and $x^3 + px^2 + qx + 1$, show that $(\underset{q}{p} - 1)^2 - \cancel{p}(\underset{q}{p} - 1) + 1 = 0$.

✓ 16. If $x^2 - px + q$ is divisible by $x - a$ and by $x - b$, show that $p = a + b$ and $q = ab$.

(*b*)

✓ 1. If $x + c$ be the H. C. F. of $x^2 + ax + b$ and $x^2 + a'x + b'$, their L. C. M. will be $x^3 + (a + a' - c)x^2 + (aa' - c^2)x + (a - c)(a' - \cancel{b}c)c$.

✓ 2. Show by the methods of examples in Art. 116, that if $ax^2 + bx + c$ and $bx^2 + cx + a$ have a common measure, then will $(a^2 - bc)^2 = (c^2 - ab)(b^2 - ac)$.

✓ 3. By the same method, prove that if $x^2 + px + q$ and $x^2 + rx + s$ have a common measure, then $\dfrac{s - q}{p - r} = \dfrac{qr - sp}{s - r}$.

✓ 4. By the same method, show that if $x^3 + px + r$ and $3x^2 + p$ have a common measure, then will $4p^3 + 27r^2 = 0$.

5. If $ax^2 - bx + c$ and $dx^3 - bx + c$ have a common factor, show that $a^3 - abd + cd^2 = 0$.

6. Find the H. C. F. of $55x^4 + 24x^3 + 1$ and $125x^4 + 24x + 1$. /

7. Find the H. C. F. of $9x^5 + 11x^3 - 2$ and $81x^5 + 11x + 4$. $3x^2 - 2x + 1$

8. Find the L. C. M. of $20x^4 + x^2 - 1$, $25x^4 + 5x^3 - x - 1$, and $50x^5 - 20x^3 + 2x$.

CHAPTER XI.

FRACTIONS.

123. Algebraic Fractions are similar to arithmetical fractions, and the same definitions, rules, and principles apply in both cases. We have seen that division may be indicated by writing the divisor below the dividend with a line between them. In the case of exact division, the work may be completed and the quotient expressed as a separate number having no connection with the divisor; thus, we may replace the indicated quotients $\frac{12}{4}$ and $\frac{4a^2b}{2ab}$ by the actual quotients 3 and $2a$. But in the case of inexact division we do not get rid of the divisor by performing the operation indicated; thus, in the indicated quotients $\frac{11}{4}$ and $\frac{ab^2+c^2}{ab}$, we get, respectively, $\frac{11}{4}=\frac{8+3}{4}=\frac{8}{4}+\frac{3}{4}=2+\frac{3}{4}$, and $\frac{ab^2+c^2}{ab}=\frac{ab^2}{ab}+\frac{c^2}{ab}=b+\frac{c^2}{ab}$, in which results the divisors still appear. The word "**fraction**" is sometimes restricted to *inexact* division, but in **both cases, and always, division is indicated**, and to both cases, therefore, the word fraction is applied as denoting the "breaking" of a quantity into parts.

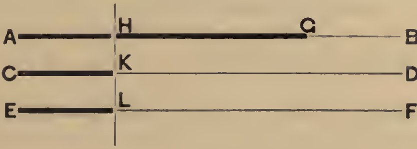
Moreover, it must be remembered that **inexact** division is really *inexact only with reference to the* **unit** *in which the dividend is expressed*, and that it may become **exact** division *by changing the dividend into an equivalent quantity expressed in units of a lower order*; thus, $\frac{\$3}{4}$ denotes inexact division in the sense that the quotient cannot be expressed as an exact number of *dollars*. But, changing the \$3 into smaller units, as 25-cent pieces, or cents, we get an exact quotient in terms of these smaller units, *e. g.*, $\frac{\$3}{4}=\frac{12 \text{ quarters}}{4}=3$ quarters, or $\frac{\$3}{4}=\frac{300 \text{ cents}}{4}=75$ cents. It is a fair inference, therefore, that, as regards *dividend*, *divisor*, and *quotient*, *whatever principles are true in exact division are true also in inexact division.*

We, therefore, define a fraction thus :—

A Fraction is an indicated quotient.

In all that relates to fractions the student should keep the idea of *division* before him.

124. There is another way of looking at a fraction, or quotient, thus : take *e. g.* $\frac{4 \text{ feet}}{2}$; here we have divided 4 ft. into *two* equal parts and taken *one* of them ; but we may obtain the same result by dividing *each unit* into *two* equal parts, and taking *four* of these parts. So, too, in $\frac{3 \text{ feet}}{4}$ there is either 3 ft. ÷4, *i. e.*, *three* ft. divided into *four* equal parts and *one* of them taken ; or, *one* ft. divided into *four* equal parts and *three* of them taken.



Thus, in the diagram, let each of the lines AB, CD, EF represent a foot in length, then *three*-fourths of *one* foot will be represented by the line AG, while *one*-fourth of *three* feet will be represented by AH + CK + EL, which is equivalent to AG.

So, generally, $\frac{a}{b}$ is $a \div b =$ *one-bth* of a, *i. e.*, a divided into b equal parts and *one* of them taken ; or, it is *one* divided into b equal parts and a of them taken, that is, $\frac{1}{b} + \frac{1}{b} + \frac{1}{b} \ldots$ to a terms.

125. From what the student has already learned of the nature of division in both Arithmetic and Algebra, he will see the truth of some important principles, which he should attentively consider :—

We have $\frac{36}{6} = 6$ and $\therefore 36 = 6 \times 6$, that is,

The dividend *is equal to the product of the divisor and quotient.* (1)

If we wish to make the indicated quotient $\frac{36}{6}$ *three* times as great, we make the *dividend* three times *as great*, or the *divisor* three times *as small*, *i. e.*, we *multiply* the dividend by 3, or *divide* the *divisor* by 3; thus, by multiplying the dividend by 3, we have $\frac{36 \times 3}{6} = 18$; or by dividing the divisor by 3, we have $\frac{36}{2} = 18$, the same result. This is, of course, true for four times, five times, six times, etc., as well as for three times; that is,

If the **dividend be increased,** *or the* **divisor be diminished,** *a certain number of times, the* **quotient is increased** *the same number of times.* (2)

If we wish to make the indicated quotient, $\frac{36}{6}$, three times as small, we may make the *dividend* three times as small, or the divisor three times as great; that is, we may *divide* the dividend by 3, or *multiply* the divisor by 3;

thus, dividing the *dividend* by 3 we get $\frac{12}{6} = 2$;

or, multiplying the *divisor* by 3 we get $\frac{36}{18} = 2$,

the same result.

This is true for four times, five times, six times, etc.: that is,

If the **dividend be diminished,** *or the* **divisor be increased,** *a certain number of times, the* **quotient is diminished** *the same number of times.* (3)

From (2) and (3) it follows that

If **both dividend and divisor** *be increased or diminished the same number of times, the quotient is not altered.* (4)

Thus, $\frac{24}{6} = \frac{24 \times 3}{6 \times 3} = \frac{72}{18} = 4$, where the dividend and divisor have been increased the same number of times. So,

$\frac{72}{18}=\frac{24\times3}{6\times3}=\frac{24}{6}=4$, when the dividend and divisor have *both* been diminished the same number of times.

We have $\frac{24}{6}=\frac{12+12}{6}=\frac{12}{6}+\frac{12}{6}=4$. And conversely we have $\frac{12}{6}+\frac{12}{6}=\frac{24}{6}=4$. That is,

If a dividend be separated into several parts (addends), the **sum of the several quotients** *will be equal to the* **complete quotient**; *and, conversely, if several dividends have the same divisor, the* **sum of the several quotients** *will be equal to the* **quotient of the sum** *of the several dividends.* (5)

It follows, also, from (4), that if the divisors are *not* the same, they can be made the same by multiplying both divisor and dividend in each case, by suitable numbers.

We have
$$\frac{36}{6}=\frac{\overset{2}{\cancel{4}}\times\overset{3}{\cancel{9}}}{\cancel{2}\times\cancel{3}}=2\times3=6.$$

But also
$$\frac{4}{2}\times\frac{9}{3}=2\times3=6\,;$$

that is
$$\frac{4}{2}\times\frac{9}{3}=\frac{4\times9}{2\times3}=\frac{36}{9}.$$
Therefore,

The product of two indicated quotients *is equal to the product of the dividends divided by the product of the divisors.* (6)

Suppose $\frac{36}{6}$ is to be divided by $\frac{9}{3}$. Dividing by 9 we get, by (3), $\frac{36}{6\times9}$; but we have used a divisor *three* times too *great*, and this quotient, $\frac{36}{6\times9}$, therefore, must be made *three* times as *great*, and the true result is $\therefore\ \frac{36\times3}{6\times9}$, by (2). Hence,

To divide by a quotient *multiply by its divisor and divide by its dividend.* (7)

N. B.—We have merely to use *numerator* for *dividend*, and *denominator* for *divisor*, and all *these propositions apply to fractions.* The numerator and denominator are the *terms* of a fraction.

The student should observe that *the separating line in a fraction operates as a vinculum.*

126. Improper Fractions.—Mixed Quantities.—When the numerator is not of lower dimensions than the denominator, the fraction is called an *improper fraction*, and it may be reduced (*i. e.* changed) to *a mixed* quantity by completing the division.

Conversely, a *mixed number* may be reduced to *an improper fraction* by reversing the process exactly as in arithmetic (Art. 125, 1), *i. e.* by multiplying the integral *quotient* by the divisor and adding the remainder to the product.

$$\frac{11}{3}=\frac{9+2}{3}=\frac{9}{3}+\frac{2}{3}=3+\frac{2}{3}=3\tfrac{2}{3} \text{ (as it is generally written).}$$

So, $\dfrac{3x+1}{x}=\dfrac{3x}{x}+\dfrac{1}{x}=3+\dfrac{1}{x}:\dfrac{4x^2+6}{2x}=\dfrac{4x^2}{2x}+\dfrac{6}{2x}=2x+\dfrac{6}{2x}\cdot$

Conversely

$$3\tfrac{2}{3}=3+\frac{2}{3}=\frac{3\times 3}{3}+\frac{2}{3}=\frac{9+2}{3}=\frac{11}{3}\cdot \quad \text{And}$$

$$3+\frac{1}{x}=\frac{3\cdot x}{x}+\frac{1}{x}=\frac{3x+1}{x}:2x+\frac{6}{2x}=\frac{2x\cdot 2x}{2x}+\frac{6}{2x}=\frac{4x^2+6}{2x}\cdot$$

Ex. 1. $\dfrac{x^2-x+1}{x-1}$; on performing the division we get quotient x and remainder 1, therefore the given fraction $=x+\dfrac{1}{x-1}\cdot$

Ex. 2. $\dfrac{x^2-4}{x-1}$; here the quotient is $x+1$ and remainder -3 $\therefore$ the fraction $=(x+1)-\dfrac{3}{x-1}\cdot$

Ex. 3. $\dfrac{5x^3-x^2+5}{5x^2+4x-1}=x-1+\dfrac{5x+4}{5x^2+4x-1}\cdot$

Ex. 4. Express $x+1+\dfrac{1}{x}$ as an improper fraction. We have $\dfrac{(x+1)\cdot x+1}{x}=\dfrac{x^2+x+1}{x}\cdot$

Ex. 5. $1+\dfrac{x-y}{x+y}=\dfrac{(x+y)\cdot 1+x-y}{x+y}=\dfrac{x+y+x-y}{x+y}=\dfrac{2x}{x+y}\cdot$

Ex. 6. $2x-1-\frac{4}{x-3}=\frac{(2x-1)(x-3)-4}{x-3}=\frac{2x^2-7x-1}{x-3}.$

Ex. 7. $x^2-3x-\frac{3x(3-x)}{x-2}=\frac{(x-2)(x^2-3x)-3x(3-x)}{x-2}$

$=\frac{x(x-3)(x-2+3)}{x-2}=\frac{x(x^2-2x-3)}{x-2}.$

EXERCISE LVII.

(a)

Reduce the following to improper fractions :—

1. $1+\frac{b}{a}$; $1-\frac{a}{b}$; $1+\frac{1}{x}$; $1-\frac{1}{x}$; $3+\frac{2}{x}$; $y+\frac{1}{x}$; $xy-\frac{1}{x}$.

$\frac{a+b}{a}$ $\frac{b-a}{b}$ $\frac{x+1}{x}$ $\frac{x-1}{x}$ $\frac{3x+2}{x}$ $\frac{xy+1}{x}$ $\frac{x^2y-1}{x}$

2. $x+1+\frac{1}{x}$; $a-1-\frac{1}{a}$; $a^2+a+\frac{1}{a^2}$; $3x^2+2x-\frac{3}{2x}$.

$\frac{x^2+x+1}{x}$ $\frac{a^2-a-1}{a}$ $\frac{a^4+a^3+1}{a^2}$ $(6x^3+4x^2-3)\div 2x$

3. $1+\frac{1+x}{x}$; $2x+\frac{x-2}{x}$; $3x^2-\frac{x^2-2}{x}$; $4ab-\frac{a^2b^2+1}{ab}$.

$2x+1\div x$ $2x^2+x-2\div x$ $3x^3-x^2+2\div x$ $3a^2b^2-1\div ab$

4. $1+\frac{1}{x-1}$; $3-\frac{2}{x-1}$; $4x^2-\frac{x^2}{x^3+1}$; $x^2-1+\frac{x+1}{x^2+1}$.

$x\div(x-1$ $3x-5\div x-1$ $4x^5+3x^2\div$ [illegible] $x^4+x\div x^2+1$

5. $1+\frac{a^2+b^2-c^2}{2ab}$; $1-\frac{a^2+b^2-c^2}{2ab}$; $x^2+x+1+\frac{2}{x-1}$.

$a^2+2ab+b^2-c^2\div 2ab$; $2ab-a^2-b^2+c^2\div 2ab$; $(x^3+1)\div x-1$

6. $x^2+3x+\frac{3x(x+3)}{x+2}$; $x-a+y+\frac{1}{x+a}$; $3x-10+\frac{41}{x+4}$.

x^3+ [illegible] $+15x)\div(x+2)$; $x^2-a^2+ax+ay+1)\div x+a$; $3x^2+2x+1\div x+4$

7. $5a-2b-\frac{3a^2-4b^2}{5a-6b}$; $a^2-ax+x^2+\frac{a^2-x^2}{a+x}$. $\frac{22a^2-40ab+16b^2}{5a-6b}$; $\frac{a^3+x^3+a^2-}{a+x}$

8. $\frac{x^4}{x^2-xy+y^2}+x^2+xy+y^2$; $\frac{a^2-ay+y^2}{x+a}+x-a+y$; $\frac{1}{x-1}+x^2+x+1$. $\frac{2x^4+x^2y^2+}{x^2-xy+y}$

$\frac{x^2+y^2+xy}{x+a}$; $\frac{x^3}{x-1}$

9. $3-\frac{6x(a+4x)}{a^2+2ax+x^4}$; $a^2+9b^2+\frac{65b^4}{a^2-9b^2}$; $4x^2+14x+\frac{98x-27}{2x-7}$.

$\frac{3a^2-2\,[illegible]\,x^2+3x^4}{a^2+2ax+x^2}$ $\frac{a^4-16b^4}{a^2-9b^2}$ $\frac{8x^3-27}{2x-7}$

10. $a^2-ab+b^2-\frac{2b^3}{a+b}$: $-\frac{2y^3}{x+y}+x^2+xy+y^2$; $2x-6x^3+18x^5-\frac{54x^7}{1+3x}$.

$\frac{a^3-b^3}{a+b}$ $\frac{x^3-y^3+2x^2y+2x}{x+y}$ $2x+6x^2$ [illegible]

(*b*)

Reduce the following to mixed quantities :—

1. $\frac{a^2+1}{a}$; $\frac{a^2-1}{a}$; $\frac{x^2y^2+y^3}{xy^2}$; $\frac{a^2+b^2}{a^2}$; $\frac{a^2b^2+a^3}{ab^2}$; $\frac{2x+3a}{x+a}$.

2. $\frac{x+1}{x-2}$; $\frac{x+3}{x-4}$; $\frac{a-3}{a-5}$; $\frac{x^2-y^2}{x^2-2y^2}$; $\frac{a^2-1}{a^2+1}$; $\frac{a^2-2}{a+1}$.

3. $\frac{x^3+1}{x-1}$; $\frac{x^2-2a^2}{x-a}$; $\frac{x^2-2ax+2a^2}{x-a}$; $\frac{1+2x+2x^2}{1+x}$.

+ $x^2+x+1+\frac{2}{x-1}$; $x+a-\frac{a^2}{x-a}$; $x-a+\frac{a^2}{x-a}$; $1+x+\frac{x^2}{1+x}$

4. $\frac{x^2-4a^2+2ax}{x+2a}$; $\frac{x^2-3x+1}{x^2-3x-1}$; $\frac{2x^2-3x+a+1}{2x^2-3x+1}$; $\frac{x^3-4a^3}{x+a}$.

+ $x-\frac{4a^2}{x+2a}$; $1+\frac{2}{x^2-3x-1}$; $1+\frac{a}{2x^2-3x+1}$; $x^2-ax+a^2-\frac{5a^3}{x+a}$

5. $\frac{x-1}{x+1}+\frac{x+1}{x-1}$; $\frac{x-3}{x-4}+\frac{x-4}{x-5}$; $\frac{x^3-3x+2}{x^3+4x^2-5}$. $1-\frac{4x^2+3x-7}{x^3+4x^2-5}$

$2-\frac{2}{x+1}+\frac{2}{x-1}$; $2+\frac{1}{x-4}+\frac{1}{x-5}$

6. $\frac{mx+b+a}{mx+c+d}$; $\frac{nx-c-d}{nx-b-d}$; $\frac{ax+m+1}{ax+m-1}$; $\frac{6x^2-7x-6\frac{2}{3}}{2x-3}$. $3x+1-\frac{}{3(2x}$

7. $\frac{ax+m}{ax+m-2}+\frac{ax+n}{ax+n-2}$; $\frac{x^2-6x-15}{x-8}+\frac{x^2-4x-11}{x-6}$. $2(x+2)-\frac{1}{x-6}+\frac{}{x}$

8. $\frac{60x^3-17x^2-4x+1}{5x^2+9x-2}$; $\frac{x^3-y^3}{x+y}$; $\frac{a^2-c^2+4b^2-4ab-(a+b+c)}{a+2b+c}$.

$12x-25+\frac{245x-49}{5x^2+9x-2}$; $x^2-xy+y^2-\frac{2y^3}{x+y}$, $a-6b-c-1+\frac{16b^2+8bc+b}{a+2b+c}$

127. Lowest Terms of Fractions.—A fraction is *reduced to lower terms* by dividing both terms by *any common factor*, and to its *lowest terms* by dividing both terms by their *highest common factor*.

This depends on the principle [Art. 125, (4)] that the value of a fraction remains unaltered when both terms are multiplied or divided by the same number. Of this principle the following proof is given :

$\frac{a}{b}$ being any fraction, we have to show that $\frac{a}{b}=\frac{ma}{mb}$, where m is any quantity.

$$\text{Quotient} \times \text{divisor} = \text{dividend} \quad [\text{Art. 125, (1)}];$$

that is,
$$\frac{a}{b} \times b = a.$$

Multiply both sides by m,

$$\therefore \quad \frac{a}{b} \times mb = ma.$$

But,

$$\frac{ma}{mb} \times mb = ma \quad \text{[Art. 125, (1)]};$$

$$\therefore \quad \frac{a}{b} \times mb = \frac{ma}{mb} \times mb.$$

Divide both sides by mb,

$$\therefore \quad \frac{a}{b} = \frac{ma}{mb}.$$

N. B.—It follows from this that *the signs of* BOTH *terms of a fraction may be changed without altering its value;* for this is equivalent to multiplying both terms by -1.

Ex. 1°. $\dfrac{axy + xy^2}{axy} = \dfrac{xy(a+y)}{xy \cdot a} = \dfrac{a+y}{a}.$

Ex. 2°. $\dfrac{cx + x^2}{a^2c + a^2x} = \dfrac{x(c+x)}{a^2(c+x)} = \dfrac{x}{a^2}.$

Ex. 3°. $\dfrac{5a^3b - 15a^2b^2}{20ab^3 + 10a^2b^2} = \dfrac{5a^2b(a-3b)}{10ab^2(2b+a)} = \dfrac{a(a-3b)}{2b(2b+a)}.$

Ex. 4°. $\dfrac{x^2 - 7x + 12}{x^2 - 9x + 20} = \dfrac{(x-3)(x-4)}{(x-5)(x-4)} = \dfrac{x-3}{x-5}.$

Ex. 5°. $\dfrac{x^2 - 12x + 35}{x^3 - 10x^2 + 31x - 30} = \dfrac{(x-5)(x-7)}{(x-5)(x^2-5x+6)} = \dfrac{x-7}{x^2-5x+6}.$

Here we find at once by inspection the factors of the numerator to be $x-5$, $x-7$; the last term (-30) of the denominator shows that $x-7$ cannot be a factor of the denominator; and $x-5$ is $\therefore$ the H. C. F.

Ex. 6°. $\dfrac{x^4 + a^2x^2 + a^4}{x^4 + ax^3 - a^3x - a^4} = \dfrac{(x^2 + a^2 + ax)(x^2 + a^2 - ax)}{(x^4 - a^4) + ax(x^2 - a^2)} =$

$$\frac{(x^2 + a^2 + ax)(x^2 + a^2 - ax)}{(x^2 - a^2)(x^2 + a^2 + ax)} = \frac{x^2 + a^2 - ax}{x^2 - a^2}.$$

Ex. 7°. $\dfrac{20x^4 + x^2 - 1}{25x^4 + 5x^2 - x - 1} = \dfrac{(5x^2-1)(4x^2+1)}{25x^4 - 1 + x(5x^2-1)} = \dfrac{4x^2+1}{5x^2+1+x}.$

EXERCISE LVIII.

Reduce the following fractions to their lowest terms:—

(*a*)

1. $\frac{ax}{bx}$; $\frac{cx}{dx}$; $\frac{a^2b}{b^3}$; $\frac{abc}{abc^2}$; $\frac{x^2y}{xy^2}$; $\frac{a^2b^2c^2}{abc^3}$; $\frac{a^2x^4}{a^4x^2}$.

2. $\frac{abx^2}{b^2x}$; $\frac{5x^4y}{20x^3y^3}$; $\frac{16a^4b^4}{20a^5b^3}$; $\frac{12a^2b^2x}{18a^4b^2y}$; $\frac{37a^2b^2c^3}{111a^3b^3c^4}$.

3. $\frac{ab^4x}{3ab^5x^2}$; $\frac{a^2x^n}{a^nx^{n-2}}$; $\frac{4x^{n-1}y^{2n}}{12x^ny^n}$; $\frac{a^{m-1}b}{b^{m-1}a}$; $\frac{ax-bx}{abx}$.

4. $\frac{ax^2}{x^2y-x^3}$; $\frac{7a^2b}{14a^3+21a^2c}$; $\frac{x^2+xy}{x^2-xy}$; $\frac{4a^2-6ab}{6a^2-4ab}$; $\frac{a^2+ab}{a^2-b^2}$.

5. $\frac{x^3+x}{x^4-1}$; $\frac{(a+x)\,y}{(a+x)^2y^2}$; $\frac{x^2-xy}{x^2y-xy^2}$; $\frac{(a+b)^2\,(a-b)}{(a-b)^2\,(a+b)^3}$; $\frac{(a+b)^2}{a^2-b^2}$. $\frac{a+b}{a-b}$

6. $\frac{a^2-b^2}{a^3-b^3}$; $\frac{x^3+1}{(x+1)^3}$; $\frac{a^3-b^3}{a^2-b^2}$; $\frac{x^2-7x}{x^2-15x+56}$; $\frac{x^2-7x+12}{x^2-9x+20}$.

$\frac{a+b}{a^2+ab+b^2}$ $\frac{x^2-x+1}{(x+1)^2}$ $\frac{a^2+ab+b^2}{a+b}$ $\frac{x}{x-8}$ $\frac{x-3}{x-5}$

7. $\frac{x^2-x-20}{x^2+2x-35}$; $\frac{x^2-2x-63}{x^2-12x+27}$; $\frac{x^2-2ax-63a^2}{x^2-12ax+27a^2}$; $\frac{x^2+7x+12}{(x+3)^2}$. $\frac{x+4}{x+3}$

$\frac{x+4}{x+7}$ $\frac{x+7}{x-3}$ $\frac{x+7a}{x-3a}$

8. $\frac{x^3-1}{x^3-3x^2+3x-1}$; $\frac{x^2+(a+b)\,x+ab}{x^2+(a+c)\,x+ac}$; $\frac{x^2-(a-b)\,x-ab}{x^2-(a+c)\,x+ac}$. $\frac{x+b}{x-c}$

$\frac{x^2+x+1}{(x-1)^2}$ $x+b$

9. $\frac{21\,(a^2+4ab-21b^2)}{28\,(a^2+2ab-15b^2)}$; $\frac{(a+b)^2-c^2}{(a+b+c)^2}$; $\frac{a^2-b}{b^2-a}$; $\frac{(a-b)^2}{(b-a)^3}$. $\frac{1}{b-a}$

$\frac{3(a+7b)}{4(a+5b)}$

10. $\frac{(x+b)^2-(x+a)^2}{(x-a)^2-(x-b)^2}$; $\frac{(c-b)\,(c-a)}{(a-b)\,(a-c)\,(b-c)}$; $\frac{(a+b)^2}{a^2-ab-2b^2}$; $\frac{a+b}{a-2b}$

$\frac{25a^4-6b^4}{5a^4-8a^2b+3b^2}$; $\frac{1+x+y+xy}{1-x^2}$. $\frac{1+y}{1-x}$

11. $\frac{a^2+2ab-c^2}{a^2+ab-ac}$; $\frac{x^2-(x+y+z)^2}{(x+y)^2-(x+z)^2}$; $\frac{6x^2-5x-6}{8x^2-2x-15}$; $\frac{(a^2-b^2)^2}{(a-b)^3}$.

$\frac{y+z}{z-y}$ $\frac{3x+2}{4x+5}$ $\frac{(a+b)^2}{a-b}$

12. $\frac{x^4+x^2y^2+y^4}{(x-y)\,(x^3-y^3)}$; $\frac{(a^3+b^3)\,(a^2+ab+b^2)}{(a^3-b^3)\,(a^2-ab+b^2)}$; $\frac{(x^4-1)}{(x^2-1)^2}$.

x^2-xy+y^2 $a+b$ $\frac{x^2+1}{x^2-1}$

(*b*)

$\frac{x-11}{4x^2-5x-5}$ $\frac{x+4}{x^2-2x+1}$

1. $\frac{x^2-16x+55}{4x^3-25x^2+20x+25}$; $\frac{6x^2-x-2}{21x^3+7x^2-3x-15}$; $\frac{x^2+x-12}{x^3-5x^2+7x-3}$.

2. $\frac{7x^2-23xy+6y^2}{5x^3-18x^2y+11xy^2-6y^3}$; $\frac{x^3-3x+2}{x^3+4x^2-5}$; $\frac{5a^5+10a^4x+5a^3x^2}{a^3x+2a^2x^2+2ax^3+x^4}$.

3. $\frac{(2y-z-x)^3-(2z-x-y)^3}{(z-x)^3-(x-y)^3}$; $\frac{abx^2+a^2xy+aby^2+b^2xy}{abx^2+a^2xy-aby^2-b^2xy}$. $\frac{9(y-z)}{z+y-2x}$

4. $\frac{12x^3+8x^2-12x+40}{24x^3-44x^2+56x-20}$; $\frac{12x^3-x^2-48x-35}{28x^3+19x^2-32x-15}$.

5. $\frac{a^4+2a^3b-2ab^3-b^4}{(a+b)^3}$; $\frac{16x^4-72x^2a^2+81a^4}{4x^2+12ax+9a^2}$; $\frac{x^5+11x^3-54}{x^5+11x+12}$.

6. $\frac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$; $\frac{ax^3-5a^2x^2-99a^3x+40a^4}{x^4-6ax^3-86a^2x^2+35a^3x}$.

7. $\frac{3a^2x^4-2ax^2-1}{4a^3x^6-2a^2x^4-3ax^2+1}$; $\frac{(a-c)(ac-b^2)+(b-c)(ab-c^2)}{(a-c)(a-b)+(b-c)^2}$.

8. $\frac{a^5-a^4b-ab^4+b^5}{a^4-a^3b-a^2b^2+ab^4}$; $\frac{x^3-6x^2-37x+210}{x^3+4x^2-47x-210}$; $\frac{55x^4+24x^3+1}{125x^4+24x+1}$.

9. $\frac{a^3+b^3+c^3-3abc}{(a+b+c)^2}$; $\frac{x^3+y^3+z^3-3xyz}{(x-y)^2+(y-z)^2+(z-x)^2}$.

10. $\frac{a(b-c)^3+b(c-a)^3+c(a-b)^3}{(a+b+c)^2\{(a-b)^3+(b-c)^3+(c-a)^3\}}$; $\frac{x^n-nx+n-1}{nx^{n+1}-(n+1)x^n+1}$.

11. $\frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{(a-b)^3+(b-c)^3+(c-a)^3}$; $\frac{x^2-xy-6y^2-xz-17yz-12z^2}{x^2-6xy+9y^2-8xz+12yz+16z^2}$.

12. $\frac{(x+y)^5-x^5-y^5}{(x+y)^4+x^4+y^4}$; $\frac{(a-b)c^4+(b-c)a^4+(c-a)b^4}{(a-b)^5+(b-c)^5+(c-a)^5}$.

128. Addition and Subtraction of Fractions.—We have

$$\frac{20}{4}=\frac{12}{4}+\frac{8}{4};$$

And conversely $\frac{12}{4}+\frac{8}{4}=\frac{12+8}{4}=\frac{20}{4}$ (Art. 125, 5).

So
$$\frac{8}{4}=\frac{5+3}{4}=\frac{5}{4}+\frac{3}{4}, \text{ etc.}$$

Also
$$\frac{a+b-c}{d}=\frac{a}{d}+\frac{b}{d}-\frac{c}{d},$$

and ∴, conversely, $\frac{a}{d}+\frac{b}{d}-\frac{c}{d}=\frac{a+b-c}{d}$.

If the denominators are not the same, we *change the fractions to equivalent fractions having a common denominator;* thus :—

$$\frac{3}{4}+\frac{2}{3}=\frac{9}{12}+\frac{8}{12}=\frac{17}{12},$$

by the principle of Art. 125, 4.

So,
$$\frac{a}{b}+\frac{a}{b^2}=\frac{ab}{b^2}+\frac{a}{b^2}=\frac{ab+a}{b^2}.$$

Generally, let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions whose sum is required, then

Quotient × divisor = dividend ;

that is,
$$\frac{a}{b}\times b=a.$$

Multiply both sides by d,

$$\therefore \quad \frac{a}{b}\times bd=ad. \qquad (1)$$

Similarly,
$$\frac{c}{d}\times d=c.$$

Multiply both sides by b,

$$\therefore \quad \frac{c}{d}\times bd=bc. \qquad (2)$$

Add (1) and (2),

$$\therefore \frac{a}{b}\times bd+\frac{c}{d}\times bd=ad+bc.$$

Divide both sides by bd,

$$\therefore \quad \frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}.$$

Hence, **to add fractions**:—

1°. Find, if necessary, the L. C. M. of the denominators; this will be the *common denominator*.

2°. Multiply the numerator of each fraction by the quotient of the *common* denominator divided by the denominator of the fraction; this will give the numerators of the new fractions.

3°. The sum of these new numerators placed over the *common* denominator will be the sum of the fractions.

Ex. 1. $\frac{a}{x}+\frac{b}{y}-\frac{c}{z}$. Here the L. C. M. of the denominators is their product xyz;

$$\therefore \quad xyz \div x = yz$$

is the multiplier for the terms of the first fraction;

$$xyz \div y = xz$$

is the multiplier for the terms of the second fraction;

and
$$xyz \div z = xy$$

is the multiplier for the terms of the third fraction;

$$\therefore \text{ the sum} = \frac{a\cdot yz}{x\cdot yz}+\frac{b\cdot xz}{y\cdot xz}-\frac{c\cdot xy}{z\cdot xy}=\frac{ayz+bxz-cxy}{xyz}.$$

Ex. 2. $\frac{x}{x+y}+\frac{xy}{x^2-y^2}-\frac{x}{x-y}$. The L. C. M. of the denominators is x^2-y^2;

$$\therefore x^2-y^2 \div (x+y) = x-y$$

is the multiplier for the terms of the first fraction.

The second fraction remains unchanged.

$$x^2-y^2 \div (x-y) = x+y$$

is the multiplier for the terms of the third fraction.

Therefore we have

$$\frac{x(x-y)+xy-x(x+y)}{x^2-y^2}=\frac{x^2-xy+xy-x^2-xy}{x^2-y^2}=-\frac{xy}{x^2-y^2}.$$

Ex. 3. $\dfrac{3}{x^2+5x+6}+\dfrac{4}{x^2+6x+8}-\dfrac{2}{x^2+7x+12}.$

By factoring the denominators we get

$$\frac{3}{(x+2)(x+3)}+\frac{4}{(x+2)(x+4)}-\frac{2}{(x+3)(x+4)},$$

in which the L. C. M. of the denominators is

$$(x+2)(x+3)(x+4).$$

Hence $x+4$ is the multiplier for the terms of 1st fraction,
$x+3$ " " " " 2d "
and $x+2$ " " " " 3d "

Hence we have

$$\frac{3(x+4)+4(x+3)-2(x+2)}{(x+2)(x+3)(x+4)}=\frac{5x+20}{(x+2)(x+3)(x+4)}=\frac{5}{(x+2)(x+3)}.$$

Ex. 4. $\dfrac{a+b}{a^2+2ab+b^2}+\dfrac{a-b}{a^2-2ab+b^2}-\dfrac{3a}{a^2-b^2}.$

Here we should *first reduce the fractions to their lowest terms*, thus getting

$$\frac{1}{a+b}+\frac{1}{a-b}-\frac{3a}{a^2-b^2},$$

which $=\dfrac{a-b+a+b-3a}{a^2-b^2}=-\dfrac{a}{a^2-b^2}.$

Ex. 5. $\dfrac{3x-2}{x+2}-\dfrac{3+2x}{x-2}-\dfrac{x^2-16x}{x^2-4}.$

Here we *first reduce the fractions to mixed quantities;* thus,

$$3-\frac{8}{x+2}-2-\frac{7}{x-2}-1+\frac{16x-4}{x^2-4}$$

$$=\frac{-8(x-2)-7(x+2)+16x-4}{x^2-4}=\frac{x-2}{x^2-4}=\frac{1}{x+2}.$$

Ex. 6. $\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1+x^4}-\dfrac{1-x}{1+x}.$

Here combining the *first* and the *last* fraction we get

$$\frac{(1+x)^2-(1-x)^2}{1-x^2}=\frac{4x}{1-x^2}.$$

Taking this result with the *second* fraction, and separating the factor $4x$, we have

$$\frac{4x}{1+x^2}+\frac{4x}{1-x^2}=4x\left(\frac{1}{1+x^2}+\frac{1}{1-x^2}\right)=\frac{8x}{1-x^4}.$$

Now take this with the third fraction, and we have

$$\frac{8x}{1-x^4}+\frac{8x}{1+x^4}=8x\left(\frac{1}{1-x^4}+\frac{1}{1+x^4}\right)=\frac{16x}{1-x^8}.$$

Ex. 7. $\dfrac{a^2-a+1}{a^2+a+1}+\dfrac{2a(a-1)^2}{a^4+a^2+1}+\dfrac{2a^2(a^2-1)^2}{a^8+a^4+1}.$

Combine the 1st and 2d fractions; the L. C. M. of the denominators is

$$a^4+a^2+1,$$

which gives $\qquad a^2-a+1$

for multiplier of the terms of the 1st fraction, and we have

$$\frac{(a^2-a+1)^2+2a(a-1)^2}{a^4+a^2+1}=\frac{a^4-a^2+1}{a^4+a^2+1}. \qquad (1)$$

Take this result with the third fraction,

$$\frac{a^4-a^2+1}{a^4+a^2+1}+\frac{2a^2(a^2-1)^2}{a^8+a^4+1},$$

in which the L. C. M. is $\qquad a^8+a^4+1,$

giving $\qquad a^4-a^2+1$

as multiplier for the terms of 1st of these fractions. Hence we have

$$\frac{(a^4-a^2+1)^2+2a^2(a^2-1)^2}{a^8+a^4+1}=\frac{a^8-a^4+1}{a^8+a^4+1}. \qquad (2)$$

The result of (2) may be written down at once by observing that the third fraction differs from the second in having a^2 written for a.

Ex. 8. $\dfrac{a^2-(b-c)^2}{(a+c)^2-b^2}+\dfrac{b^2-(c-a)^2}{(b+a)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$.

Here *the principle of symmetry* will help us to shorten the work: on reducing the 1st fraction to its *lowest terms* by dividing by $a+c-b$ we get

$$\frac{a+b-c}{a+b+c},$$

and since the whole expression is symmetrical in a, b, c, $a+b+c$ will be the denominator of all the reduced fractions, and the numerators may be written down, by symmetry, from $a+b-c$. Hence we have

$$\frac{(a+b-c)+(b+c-a)+(c+a-b)}{a+b+c}=\frac{a+b+c}{a+b+c}=1.$$

129. By observing examples 4, 5, 6, 7, 8, the student will see that the work may often be simplified by attending to the following points:—

1°. *The several fractions should be reduced to their lowest terms.*

2°. *Improper fractions may sometimes be reduced to mixed numbers and the results combined.*

3°. *The fractions may be combined in pairs; or, two of them may be combined and the result combined with a third, etc.*

4°. *When two fractions have a common numerator, this may be taken out, and the result multiplied by it.* (See work in Ex. 6.)

5°. *The principle of symmetry may sometimes be applied.*

EXERCISE LIX.

(*a*)

Combine the following fractions:—

1. $\dfrac{1}{x}+\dfrac{2}{x}$; $\dfrac{3}{x}-\dfrac{2}{x}$; $\dfrac{2a}{x}-\dfrac{a}{x}$; $\dfrac{x}{a+b}+\dfrac{y}{a+b}$; $\dfrac{5\frac{1}{2}}{y}+\dfrac{6\frac{1}{2}}{y}$.

2. $\dfrac{a+b}{5}+\dfrac{3(a+b)}{5}$; $\dfrac{5(a-b)}{7}-\dfrac{3(a-b)}{7}$; $\dfrac{a-b}{b}-\dfrac{a+b}{b}$.

3. $\frac{1}{a}+\frac{1}{b};\quad \frac{1}{a}-\frac{1}{b};\quad \frac{a}{x}-\frac{x}{a};\quad \frac{1}{ax}+\frac{1}{bx};\quad \frac{a^2}{x^2}+\frac{a}{x};\quad \frac{a^2}{x}-\frac{a}{x}.$

4. $\frac{x-3}{2}+\frac{x-4}{3};\quad \frac{x+3}{2}-\frac{x-6}{3};\quad \frac{a}{5}+\frac{a-1}{7};\quad \frac{a^2}{5b}-\frac{1}{10b^2}.$

5. $\frac{4x-1}{3}-\frac{3x-2}{4};\quad \frac{x-a}{a}-\frac{x-b}{b};\quad \frac{1}{a}+\frac{1}{b}+\frac{1}{c};\quad \frac{ab}{c}-\frac{c}{ab}.$

6. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca};\quad \frac{x}{yz}+\frac{y}{zx}+\frac{z}{xy};\quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c};\quad \frac{1}{9}+\frac{2}{x}-\frac{3}{x^2}.$

7. $\frac{a}{2x}+\frac{b}{3x}+\frac{c}{4x};\quad \frac{x}{y}+\frac{x}{z}+\frac{x}{a};\quad \frac{a}{x^n}+\frac{b}{x^{n-1}}+\frac{c}{x^{n-2}};\quad \frac{x}{ab}-\frac{ab}{x}.$

8. $\frac{1}{x-1}+\frac{1}{x+1};\quad \frac{1}{a-b}-\frac{1}{a+b};\quad \frac{x}{x-a}+\frac{x}{x+a};\quad \frac{a}{a-x}-\frac{x}{a+x}.$

9. $\frac{1}{x-4}-\frac{1}{x+7};\quad \frac{1}{a+b}-\frac{1}{b+c};\quad \frac{2}{x+2}-\frac{1}{x+3};\quad \frac{a-1}{a+b}-\frac{1-a}{a-b}.$

10. $\frac{x+1}{x-1}+\frac{x-1}{x+1};\quad \frac{x-a}{x+a}+\frac{x+a}{x-a};\quad \frac{x-7}{x-8}+\frac{x-8}{x-9};\quad \frac{2x+3}{2x-3}-\frac{2x-3}{2x+3}.$

11. $\frac{x+y}{4y}-\frac{x}{x+y};\quad \frac{6}{(x-1)(x-2)}-\frac{3}{(x-1)(x-2)(x+4)};$

$\frac{x-a}{x^2-a^2}+\frac{a}{x+a};\quad \frac{1}{n-1-(n-1)^x}-\frac{1}{n+1+(n+1)^x}.$

12. $\frac{x+1}{x^2+x+1}+\frac{x-1}{x^2-x+1};\quad \frac{x+a}{x^2-ax+a^2}-\frac{1}{x+a};\quad \frac{1}{x-a}-\frac{x-a}{x^2+ax+a^2}.$ $\frac{3c}{x^3}$

$+2x^3 \div x^4+x^2+1$ $\quad 3ax \div x^3+a^3$

13. $\frac{2}{a+x}+\frac{2}{a-x}-\frac{2a}{a^2-x^2};\quad \frac{a}{a(a-x)}+\frac{1}{a-x};\quad \frac{1}{a-x}-\frac{x}{(a-x)a}.$ $\frac{1}{a}$

$-2 \div (a^2-x^2)$ $\quad 2 \div (a-x)$

14. $\frac{x}{x-1}-\frac{x+1}{x^2-1};\quad \frac{a}{(1-a)^2}+\frac{a^2}{(1-a)^3};\quad \frac{x+a}{x-a}+\frac{x-a}{x+a}-\frac{2ax}{x^2-a^2}.$ $\frac{2(x}{x}$

1 $\quad a \div (1-a^3)$

15. $\frac{x-y}{x+y}+\frac{x^2-2y^2}{x^2-y^2};\quad \frac{1}{(x-y)^2}-\frac{1}{(x+y)^2};\quad \frac{1}{a+b}+\frac{3ab}{a^3+b^3};$ $\frac{a}{a^2-}$

$\frac{2x^2-2xy-y^2}{x^2-y^2}$ $\quad 4xy \div (x^2-y^2)^2$

$\frac{1}{a-b}-\frac{3ab}{a^3-b^3};\quad \frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}.$ $x^{2n}+2$

$(a-b) \div (a^2+ab+b^2)$

16. $\frac{1}{x^2-5x+6}+\frac{1}{x^2-6x+8};$ $\quad \frac{2x-7}{(x-2)(x-3)(x-4)}$

$\frac{a+x}{a^2+ax+x^2}+\frac{a-x}{a^2-ax+x^2}+\frac{2x^2}{a^4+a^2x^2+x^4}.$ $\quad \frac{2(a+x)}{a^2+a\quad+x}$

(*b*)

Combine the following :—

1. $\dfrac{5}{1+2x}-\dfrac{3x}{1-2x}-\dfrac{4-13x}{1-4x^2};\quad \dfrac{1}{2x+y}+\dfrac{1}{2x-y}-\dfrac{3x}{4x^2-y^2}.$

$(1-6x^2)\div(1-4x^2)\qquad x\div(4x^2-y^2)$

2. $\dfrac{2a}{2a+3b}+\dfrac{3b}{2a-3b}-\dfrac{8b^2}{4a^2-9b^2};\quad \dfrac{b}{a^2+ab}-\dfrac{a}{ab+b^2}+\dfrac{(a+b)^2}{a^2b-ab^2}.$

$(4a^2+b^2)\div 4a^2-9b^2\qquad 4\div(a-b)$

3. $\dfrac{1}{x-3}+\dfrac{x-3}{x^2+3x+9}+\dfrac{3x-2x^2}{x^3-27};\quad \dfrac{x-2}{x^2-2x+4}+\dfrac{1}{x+2}+\dfrac{2x}{x^3+8}.$

$18\div(x^3-27)\qquad 2x^2\div(x^3+8)$

4. $\dfrac{x+4}{2x+10}+\dfrac{x+3}{3x-15}+\dfrac{x+2}{4x^2-100}+\dfrac{1}{2};\quad \dfrac{1}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{x+3}{2(x^2+1)}.$

$(16x^2+29x-204)\div 12(x-5)(x+5)\qquad (x+3)\div x^4-1$

5. $\dfrac{3}{x-1}-\dfrac{2}{x+4}-\dfrac{3}{x^2+3x-4};\quad \dfrac{3x}{(x+y)^2}-\dfrac{x+2y}{x^2-y^2}+\dfrac{3y}{(x-y)^2}.$

$(x+11)\div(x-1)(x+4)\qquad (2x^3-5x^2y+10xy^2+5y^3)\div(x^2-y^2)^2$

6. $\dfrac{1}{a^2-(b-c)^2}+\dfrac{1}{b^2-(a-c)^2}+\dfrac{1}{c^2-(a-b)^2};\quad \dfrac{a+b+c}{(a+b-c)(a+c-b)(b+c-a)}$

$\dfrac{1}{x^2-1}-\dfrac{2}{(x+1)(x+3)}+\dfrac{1}{(x+3)(x-1)}.\quad \dfrac{0}{(x-1)(x+1)(x+3)}$

7. $\dfrac{x}{(x+y)(x+2y)}+\dfrac{2y}{(x+y)(x+3y)}+\dfrac{x}{(x+2y)(x+3y)}-\dfrac{3}{x+3y};$

$\dfrac{1}{x^2-9}-\dfrac{1}{x^2+9}-\dfrac{18}{x^4+81}.\qquad \dfrac{-1}{x+3y}\qquad \dfrac{2916}{x^8-6561}$

8. $\dfrac{1}{x-a}-\dfrac{1}{x+a}-\dfrac{2a}{x^2+a^2}-\dfrac{4a^3}{x^4+a^4};\quad \dfrac{ab}{(a-b)(b-c)}+\dfrac{ac}{(a-c)(c-b)}.$

$\dfrac{8a^7}{x^8-a^8}$

9. $\dfrac{a+3b}{4(a+b)(a+2b)}+\dfrac{a+2b}{(a+b)(a+3b)}-\dfrac{a+b}{4(a+2b)(a+3b)}.$

$\dfrac{1}{a+b}$

10. $\dfrac{x}{x^2+5x+6}+\dfrac{15}{x^2+9x+14}-\dfrac{12}{x^2+10x+21};$

$\dfrac{a+b-c-d}{a-b+c-d}-\dfrac{a-b-c+d}{a+b+c+d}.$

11. $\dfrac{xy}{bc}+\dfrac{(x-b)(y-b)}{b(b-c)}+\dfrac{(x-c)(y-c)}{c(c-b)};$

$\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}.$

12. $\frac{a+x}{x(x-y)(x-z)} + \frac{a+y}{y(y-x)(y-z)} + \frac{a+z}{z(z-x)(z-y)}$; $\frac{a}{xyz}$

$\frac{x}{1-x} - \frac{x^2}{(1-x)^2} + \frac{x^3}{(1-x)^3}$. $\frac{x(1-3x+3x^2)}{(1-x)^3}$

13. $\frac{3}{8(1-x)} + \frac{1}{8(1+x)} - \frac{1-x}{4(1+x^2)}$; $\frac{a+c}{(a-b)(x-a)} - \frac{b+c}{(a-b)(x-b)}$.

$\frac{1+2x+3x^2}{4(1-x^4)}$

14. $\frac{a^m+b^m}{a^m-b^m} - \frac{a^m-b^m}{a^m+b^m}$; $\frac{y^n}{(y-z)^n} - \frac{y^{n-1}}{(y-z)^{n-1}}$. $\frac{4a^m b^m}{a^{2m}-b^{2m}}$;

15. $\frac{b}{a^{x-y}} + \frac{c}{a^{x-z}} + \frac{d}{a^{x-u}} - \frac{c}{a^x}$; $\frac{3}{x^2-1} + \frac{4}{2x+1} + \frac{4x+2}{2x^2+3x+1}$.

$(a^y b + a^z c + a^u d - c) \div a^x$; $(8x^2+11x-3) \div (x-1)(x+1)(2x+1)$

16. $\frac{3}{8(a-x)} + \frac{1}{8(a+x)} - \frac{a-x}{4(a^2+x^2)}$; $\frac{x^2+y^2}{x^2-y^2} + \frac{x}{x+y} + \frac{y}{y-x}$.

$a(a^2+2ax+3x^2) \div 4(a^4-x^4)$

17. $\frac{1-2x}{3(x^2-x+1)} + \frac{1+x}{2(x^2+1)} + \frac{1}{6(x+1)}$;

$\frac{x-a}{x-b} + \frac{x-b}{x-a} - \frac{(b-a)^2}{(x-a)(x-b)}$. 2

18. $\frac{1}{2x+2} - \frac{4}{x+2} + \frac{9}{2(x+3)} - \frac{x-1}{x^2+5x+6}$.

(*c*)

Simplify the following :—

1. $\frac{a^2+ab}{ab-b^2} - \frac{a(a-b)}{(a+b)b} - \frac{2ab}{a^2-b^2}$; $\frac{b^2}{a^3-b^3} + \frac{a^2b^2}{b^6-a^6}$.

2. $\frac{1}{1-x} - \frac{1}{1+x} - \frac{2x}{1+x^2} - \frac{4x^3}{1+x^4} - \frac{8x^7}{1+x^8}$; $\frac{p}{x-a} + \frac{q}{x-b} - \frac{p+q}{x-c}$.

$16x^{15} \div$

3. $\frac{1-x}{1-a} - \frac{1-x}{1-a^2} + \frac{1-x}{1-a+a^2-a^3}$; $\frac{a-c}{(a-b)(x-a)} - \frac{b-c}{(b-a)(b-x)}$.

$\frac{x-c}{(x-a)(x-b)}$

4. $\frac{4}{x-1} - \frac{9}{x-3} + \frac{6}{x-5} - \frac{1}{x-7}$;

$24(x-9) \div (x-1)(x-3)(x-5)(x-7)$

$\frac{2}{x-1} - \frac{3}{x+2} - \frac{4}{7(x-3)} - \frac{11}{7(x+4)}$.

5. $\frac{1}{(a-b)(a-c)} + \frac{1}{(b-a)(b-c)} + \frac{1}{(c-a)(c-b)}$. 0

6. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-a)(b-c)}+\dfrac{c}{(c-a)(c-b)}.$ 0

7. $\dfrac{a^2}{(a-b)(b-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}.$ 1

8. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-a)(b-c)}-\dfrac{c^3}{(a-c)(c-b)}.$ $a+b+c$

9. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}.$ 0

10. $\dfrac{x^2yz}{(x-y)(x-z)}+\dfrac{y^2zx}{(y-z)(y-x)}+\dfrac{z^2xy}{(z-x)(z-y)}.$ 0

11. $\dfrac{1+a}{(a-b)(a-c)}+\dfrac{1+b}{(b-c)(b-a)}+\dfrac{1+c}{(c-a)(c-b)}.$ 0

12. $\dfrac{p-a}{(p-q)(p-r)}+\dfrac{q-a}{(q-r)(q-p)}+\dfrac{r-a}{(r-p)(r-q)}.$ 0

13. $\dfrac{a^2}{(a^2-b^2)(a^2-c^2)}+\dfrac{b^2}{(b^2-c^2)(b^2-a^2)}+\dfrac{c^2}{(c^2-a^2)(c^2-b^2)}.$ 0

14. $\dfrac{1}{(b+c-2a)(c+a-2b)}+\dfrac{1}{(c+a-2b)(a+b-2c)}$

$\quad+\dfrac{1}{(a+b-2c)(b+c-2a)}.$ 0

15. $\dfrac{x(y+z}{(x-y)(z-x)}+\dfrac{y(z+x)}{(y-z)(x-y)}+\dfrac{z(x+y)}{(z-x)(y-z)}.$ 1

16. $x^3\left(\dfrac{x^3-2y^3}{x^3+y^3}\right)^3 \mp y^3\left(\dfrac{2x^3-y^3}{x^3+y^3}\right)^3.$ x^3-y^3

17. $\dfrac{a^{2x}+a^xb+b^2}{a^{4x}-a^{3x}b+a^{2x}b^2-a^xb^3+b^4}-\dfrac{1}{a^{2x}-a^xb+b^2}.$

$a^xb(a^{2x}+b^2)\div(a^{2x}-a^xb+b^2)\ a^{4x}-a^{3x}+a^2$ [illegible] $-a^xb^3+b^4$

18. $\dfrac{a^{x+y+z}-a^{x-y+z}}{a^{x-y-z}-a^{x-y+z}}-\dfrac{a^{x+y-z}-a^{x-y+z}}{a^{x+y+z}+a^{x+y-z}}.$ $\dfrac{a^{4z}(a^{2y}-\frac{1}{a^{2y}})+a^{2z}(a^{2y}+\text{[illegible]})-(a^{2z}+1)}{1-a^{6z}}$

130. Multiplication of Fractions.—

i. To multiply a fraction **by a whole number.**

ii. To multiply a fraction **by a fraction.**

i. If $\frac{24}{8}$ is to be multiplied by 4, we have (Art. 125, 2),

$$\frac{24}{8} \times 4 = \frac{96}{8} = 12.$$

Or,
$$\frac{24 \times 4}{8} = \frac{24}{2} = 12.$$

So, also,
$$\frac{a^3b^2}{b^4} \times b^2 = \frac{a^3b^2 \cdot b^2}{b^4} = \frac{a^3b^4}{b^4} = a^3.$$

Or,
$$\frac{a^3b^2}{b^4} \times b^2 = \frac{a^3b^2 \cdot b^2}{b^4} = \frac{a^3b^2}{b^2} = a^3.$$

That is, in both the arithmetical and the algebraic examples we multiplied the numerator or *divided the denominator.*

ii. Generally, if $\frac{a}{b}$ is to be multiplied by $\frac{c}{d}$, this means that the product of the *two dividends* (a and c) is to be divided by the product of the *two divisors.* Thus,

$$\frac{36}{6} = \frac{4 \times 9}{2 \times 3} = \frac{4}{2} \times \frac{9}{3};$$

and, conversely,
$$\frac{4}{2} \times \frac{9}{3} = \frac{4 \times 9}{2 \times 3} = \frac{36}{6}.$$

The following is a *general* proof :—

$\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions, then

Quotient × Divisor = Dividend.

Or,
$$\frac{a}{b} \times b = a \quad \text{(Art. 125, 1)};$$

similarly,
$$\frac{c}{d} \times d = c.$$

Multiply these equals,

$$\therefore \frac{a}{b} \times \frac{c}{d} \times bd = ac.$$

Divide both sides by bd,

$$\therefore \qquad \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}.$$

Case i is included in this by putting $d = 1$. Therefore,

To multiply two fractions:—

Multiply the numerators for a new numerator and the denominators for a new denominator.

131. Division of Fractions.—

i. To divide a fraction **by a whole number.**

ii. To divide a fraction **by a fraction.**

i. If we have $\frac{12}{5}$ to divide by 4, we have (Art. 125, 3) either

$$4\,)\,\frac{12}{5} = \frac{3}{5},$$

Or,
$$\frac{12}{5} \div 4 = \frac{12}{5 \times 4} = \frac{12}{20} = \frac{3}{5}.$$

So,
$$\frac{a^2}{b^2} \div a = a\,)\,\frac{a^2}{b^2} = \frac{a}{b^2},$$

Or,
$$\frac{a^2}{b^2} \div a = \frac{a^2}{a \cdot b^2} = \frac{a}{b^2}.$$

That is, to divide a fraction by a whole number, we divide the numerator when this can be done exactly, or multiply the denominator.

ii. In the case of $\frac{36}{3} \div \frac{12}{4}$, we are to divide one quotient by another, *i. e.*, $\frac{36}{3}$ is to be divided *not* by 12, but by *one-fourth* of 12, that is, 3. Divide by 12 and we get $\frac{36}{3 \times 12}$; but $\frac{36}{3 \times 3}$ is the true quotient, and this is got from $\frac{36}{3 \times 12}$ by multiplying it by 4; thus,

$$\frac{36 \times 4}{3 \times 12} = \frac{36}{3 \times 3};$$

thus the quotient is obtained by *inverting* the divisor and proceeding as in multiplication.

The following is a general proof:—

$\frac{a}{b}$ and $\frac{c}{d}$ being any fractions, it is required to divide $\frac{a}{b}$ by $\frac{c}{d}$. We have

$$\text{Quotient} \times \text{Divisor} = \text{Dividend};$$

that is
$$\left(\frac{a}{b}\right) \times b = a.$$

Multiply both sides by d,

$$\therefore \quad \left(\frac{a}{b}\right) \times bd = ad. \qquad (1)$$

Similarly,
$$\left(\frac{c}{d}\right) \times d = c.$$

Multiply both sides by b,

$$\therefore \quad \left(\frac{c}{d}\right) \times bd = bc. \qquad (2)$$

Dividing (1) by (2) bd cancels,

$$\therefore \quad \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}.$$

Hence, **to divide one fraction by another**:—

Invert the divisor and proceed as in multiplication.

N. B.—In multiplication and division of fractions the student should always first *indicate the operations to be performed, and then cancel all factors common to numerator and denominator.*

Ex. 1. Simplify $\frac{a^2+x^2}{a^2-x^2} \times \frac{a^3+x^3}{a^4-x^4}$.

By resolving into factors we have

$$\frac{a^2+x^2}{(a+x)(a-x)} \times \frac{(a+x)(a^2-ax+x^2)}{(a^2+x^2)(a^2-x^2)};$$

and this, by striking out the common factors,

$$= \frac{1}{a-x} \times \frac{a^2-ax+x^2}{a^2-x^2} = \frac{a^2-ax+x^2}{(a-x)(a^2-x^2)}.$$

Ex. 2. $$\frac{2x^3+5x+2}{x^2-4} \div \frac{2x^2+9x+4}{x^2+4x};$$

by factoring we find this

$$= \frac{(2x+1)(x+2)}{(x-2)(x+2)} \div \frac{(2x+1)(x+4)}{x(x+4)};$$

strike out now the H. C. F.

$$= \frac{2x+1}{x-2} \div \frac{2x+1}{x};$$

now invert the divisor

$$= \frac{2x+1}{x-2} \times \frac{x}{2x+1} = \frac{x}{x-2}.$$

Ex. 3. $$\frac{4x^2+x-14}{6xy-14y} \times \frac{4x^2}{x^2-4} \times \frac{x-2}{4x-7} \div \frac{2x^2+4x}{3x^2-x-14}$$

$$= \frac{(4x-7)(x+2)}{2y(3x-7)} \times \frac{4x^2}{(x+2)(x-2)} \times \frac{x-2}{4x-7} \div \frac{2x(x+2)}{(3x-7)(x+2)}$$

$$= \frac{(4x-7)(x+2)}{2y(3x-7)} \times \frac{4x^2}{(x+2)(x-2)} \times \frac{x-2}{4x-7} \times \frac{(3x-7)(x+2)}{2x(x+2)}$$

$$= \frac{1}{2y} \times \frac{4x^2}{1} \times 1 \times \frac{1}{2x} = \frac{4x^2}{4xy} = \frac{x}{y}.$$

N. B.—Since a fraction indicates division, the RULE OF SIGNS *holds as in ordinary division.*

EXERCISE LX.

(*a*)

Perform the following operations :—

1. $\frac{a}{b} \times c$; $\frac{a}{b} \div c$; $\frac{a}{b} \times \frac{1}{c}$; $\frac{a}{b} \div \frac{1}{c}$; $\frac{a}{b} \times \frac{p}{q}$; $\frac{a}{b} \div \frac{p}{q}$.

2. $\frac{4a^2}{5b} \div a$; $\frac{4a}{5b} \times \frac{2}{5}$; $\frac{a}{b} \times ax$; $\frac{b}{c} \times cx$; $\frac{a}{x} \div ax$; $\frac{3a^2}{b} \times \frac{3b}{a}$.

3. $\frac{4}{5} \times \frac{2a^2}{b^2}$; $\frac{4}{9} \div \frac{3a^2}{2b^2}$; $\frac{4x^3}{a^2} \times \frac{2a}{3x^2}$; $\frac{4x^3}{a^2} \div \frac{2a}{x^2}$; $\frac{x}{y} \times \frac{y}{z} \times \frac{z}{x}$.

4. $\frac{a}{b}\div\frac{b}{c}\times\frac{c}{a}$; $\frac{a}{b}\times\frac{b}{c}\div\frac{c}{a}$; $\frac{1}{a}\div\frac{1}{b}$; $\frac{x^2}{a^2}\div\frac{a^2}{x^2}$; $1\div\frac{a^2}{b^2}$; $1\div\frac{1}{b+x}$.

5. $1\div\frac{a-b}{a+b}$; $\frac{6a^5}{35x^3}\times\frac{14a^3}{9x^3}$; $\frac{6a^5}{14x^3}\div\frac{9a^3}{35x^4}$; $\frac{a^2}{bc}\times\frac{b^2}{ac}\times\frac{c^2}{ab}$.

6. $\frac{a+3}{a+4}\times\frac{a-3}{a-4}$; $\frac{a^2+4}{a^2-5}\div\frac{a^2+4}{a^2-5}$; $\frac{x^2-y^2}{x^2+y^2}\div(x+y)$; $\frac{x+3}{x+1}\times(x^2+5x+4)$; $\frac{1}{x^n}\div\frac{1}{x^{n-1}}$.

$\frac{a^2-9}{a^2-16}$; 1; $(x+3)(x+1)$ $\frac{1}{x}$

7. $\frac{x+3}{x+4}\div(x^2+7x+12)$; $\frac{x^2-y^2}{(x+y)^2}\div\frac{x-y}{x+y}$; $\frac{a}{x-1}\div\frac{b}{1-x}$; $(xy-yz)\div xyz$; $5a^{3m}b\div 3a^{5m}c$.

$\frac{1}{(x+4)^2}$; 1; $\frac{1}{z}-\frac{1}{x}$; $\frac{5b}{3a^{2m}c}$

8. $\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\times xyz$; $\left(\frac{2}{ab}+\frac{2}{bc}+\frac{2}{ca}\right)\times abc$; $\left(\frac{1}{a^2bc}+\frac{1}{b^2ca}+\frac{1}{c^2ab}\right)\times a^2b^2c^2$.

$xy+yz+zx$, 2

$ab+bc+ca$

9. $\left(\frac{1-x}{1+x}+\frac{1+x}{1-x}\right)(1-x^2)$; $\left(\frac{a+b}{a-b}-\frac{a-b}{a+b}\right)(a^2-b^2)$; $\frac{a}{x^2-a^2}\times\frac{x+a}{a^2}$; $\frac{(a+b+c)^2-d^2}{(a+b+c+d)^2}\div\frac{m(a+b+c-d)}{n(a+b+c+d)}$.

$2+2x^2$; $4a$ $\frac{1}{a(x-a)}$

10. $\left(\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\right)\div abc$; $\left(\frac{a-b}{a+b}+\frac{a+b}{a-b}\right)\div(a^2-b^2)$; $\frac{x+1}{x-1}\times\frac{x^2-1}{(x+1)^2}$; $\left(\frac{1+b}{1-d}\right)^3\div\left(\frac{1+b}{1-d}\right)^4$.

$\frac{1}{a^2}+\frac{1}{b^2}+$ $\frac{2(a^2+b^2)}{(a^2-b^2)^2}$; 1; $\frac{[illegible]}{1-d}$

11. $\left\{\frac{x+3}{(x-3)(x-1)}-\frac{x+4}{(x-4)(x-1)}\right\}\times(x-1)(x-3)(x-4)$; $\frac{x^2}{(x+y)^2}\times\frac{(x+y)^2}{y^2}$; $\frac{a^{5x}b^{7y}c^{3z}}{a^{2x}b^{3y}c^{4z}}\div abc$.

$-2x$ $\frac{x^2}{y^2}$; $\frac{a^{3x-1}b^{4y-1}}{c^{z+1}}$

12. $\frac{a}{x-b}\div\frac{x-b}{x}$; $\frac{x+3}{x(x-3)}\times\frac{x^2}{x+3}$; $\frac{x-1}{x+1}\div\frac{1}{x+1}$; $\frac{(x-a)^2}{a^2}\div\frac{1}{(x-a)^3}$.

$\frac{ax}{(x-b)^2}$; $\frac{x}{x-3}$; x $(x-a)^5\div a^2$

13. $\frac{x+3}{x-3}\div\frac{(x-3)^2}{(x+3)^2}$; $\frac{a}{a+b}\times\frac{a}{a-b}\times\frac{a^2-b^2}{ab}$; $\frac{a^2}{a^2-x^2}\times\frac{a^2+ax+x^2}{a(a+x)}$.

$\frac{x+3}{x-3}$ $\frac{a}{b}$; $\frac{a(a^2+ax+x^2)}{(a-x)(a+x)^2}$

14. $\dfrac{x^3-y^3}{x^3+y^3} \div \dfrac{x-y}{x^2-xy+y^2}$; $\dfrac{1}{x+a} \times \dfrac{x+a}{x^2+a^2} \div \dfrac{x^3+a^3}{x^2+a^2}$; $\dfrac{x^2+xy+y^2}{x+y}$; x^3

$\dfrac{a^2+b^2}{a^2-b^2} \times \dfrac{a^3+b^3}{a^4-b^4}$; $\dfrac{1+b^{2n}}{1-b^{2n}} \times \dfrac{1+b^{3n}}{1-b^{4n}}$. $\dfrac{a^2-ab+b^2}{(a+b)(a-b)^2}$; $\dfrac{1-b^n+b^{2}}{(1+b^n)(1-b}$

15. $\dfrac{x^2-1}{x^2+1} \times \dfrac{x^4-1}{x^3-1}$; $\dfrac{x+1}{x+2} \times \dfrac{x+2}{x+3} \times \dfrac{x+3}{x+1}$; $\dfrac{(x^2-1)(x+1)}{x^2+x+1}$; 1;

$\dfrac{(x-a)(x+b)}{(x+c)^2} \times \dfrac{x^2-c^2}{x^2-a^2}$. $\dfrac{(x+b)(x-c)}{(x+a)(x+c)}$

16. $\dfrac{(x-2a)(x-3b)}{(x-2c)^2} \times \dfrac{x-2c}{x-2a}$: $\dfrac{(x+a)^2(x-b)^2}{(x+b)^2(x-c)^3} \times \dfrac{(x+b)(x-c)^5}{(x+a)^3(x-b)^3}$.

$(x-3b) \div (x-2c)$; $(x-c)^2 \div (x+a)(x-b)(x+b)$

17. $\dfrac{x^2+(a+b)x+ab}{x^2-c^2} \div \dfrac{x+a}{x+c}$; $\dfrac{x^2+5x+6}{x+4} \times \dfrac{x^2+7x+12}{x^2+6x+9}$; $\dfrac{x+b}{x-c}$

$\left(1+\dfrac{1}{x}\right)\left(\dfrac{1}{x}-1\right)$. $(x+2)$; $\dfrac{1}{x^2}-1$

(*b*)

Simplify the following :—

1. $\dfrac{x+y}{(x-y)^3} \times \dfrac{x^3-y^3}{x^3+y^3} \times \dfrac{(x-y)^2+xy}{(x+y)^2-xy}$; $\left(\dfrac{7a^2}{3b}\right)^m \div \left(\dfrac{14a}{15b^3}\right)^m$. $\dfrac{x}{(x-y)^2}$; $\Big($

2. $\dfrac{a^3-8b^3}{a^2-4b^2} \times \dfrac{a+2b}{a^2+2ab+4b^2}$; $\dfrac{x^2-1}{x^2-3x-10} \div \dfrac{x^2-12x+35}{x^2+3x+2}$. 1; $\dfrac{(x^2-1)(x+1)}{(x-7)(x-5}$

3. $\dfrac{a^3+x^3+3ax(a+x)}{x^3-a^3} \div \dfrac{(a+x)^2}{a^2+x^2+ax}$; $\left(a+b+\dfrac{a^2}{b}\right) \div \left(a+b+\dfrac{b^2}{a}\right)$.

$(x+a) \div (x-a)$ $a \div b$

4. $\left(a-b+\dfrac{b^2}{a+b}\right) \div \left(a+b+\dfrac{b^2}{a-b}\right)$; $\left(\dfrac{3ab}{5cd}\right)^4 \cdot \left(\dfrac{5c}{6a}\right)^3 \cdot \left(\dfrac{4b}{3d}\right)^2$. $\dfrac{2ab^6}{15cd^6}$

$(a-b) \div (a+b)$

5. $\dfrac{x^2+x-2}{x^2-x-20} \times \dfrac{x^2+5x+4}{x^2-4} \div \left(\dfrac{x^2+3x+2}{x^2-2x-15} \times \dfrac{x+3}{x^2}\right)$; $\dfrac{x^2(x-1)}{(x^2-4)}$

$\left(\dfrac{4x^n}{y^p}\right)^m \cdot \left(\dfrac{25y^{p+1}}{x^{n-1}}\right)^m$; $\left(\dfrac{a^3}{b^3}+\dfrac{b^3}{a^3}\right) \div \left(\dfrac{a}{b}+\dfrac{b}{a}\right)$. $100xy)^m$ $\dfrac{a^2}{b^2}+\dfrac{b^2}{a^2}-$

6. $\dfrac{x^2+12ax-13a^2}{c^2(x^2+17ax-38a^2)} \times \dfrac{x+19a}{x-a} \div \dfrac{b^2}{a^2c}$; $\dfrac{(2ab)^5}{(3b)^3} \times \dfrac{(3ab)^2 \cdot (5a)^4}{(4ab)^6}$.

$\dfrac{a^2(x+13a)}{b^2(x-2a)}$ $\dfrac{625a^{15}}{384c^2}$

7. $\dfrac{(a+b)^2-c^2}{a^2-(b+c)^2} \div \dfrac{(b-c)^2-a^2}{(a-c)^2-b^2}$; $\dfrac{(x^{a+b})^2 \cdot (x^{b+c})^2 \cdot (x^{c+a})^2}{(x^a \cdot x^b \cdot x^c)^4}$.

$\dfrac{a+b-c}{b-c-a}$; 1

8. $\left(x+\dfrac{xy}{x-y}\right)\left(x-\dfrac{xy}{x+y}\right) \div \dfrac{x^2+y^2}{x^2-y^2}$; $\left(\dfrac{3a^2b^3c^4}{5d^5e^6f^7}\right)^m \div \left(\dfrac{9a^4b^2c}{25d^6e^7f^2}\right)^m$.

$x^4 \div (x^2+y^2)$;

9. $\dfrac{x^4-a^4}{x^6+a^6} \times \dfrac{x^3+a^3}{x^4+2x^2a^2+a^4} \div \dfrac{x^3-a^3}{x^4-a^2x^2+a^4}$;

$\dfrac{3x-1}{2x^2-11x+12} \times \dfrac{2x-3}{9x^2-9x+2}$.

10. $\dfrac{x^2-1}{y+1} \times \dfrac{y^2-1}{x^2+x} \times \left(1+\dfrac{x}{1-x}\right)$; $\dfrac{(x^{p+2q})^3 \cdot (x^{q+2r})^3 \cdot (x^{r+2p})^3}{(x^p \cdot x^q \cdot x^r)^9}$.

11. $\left(x^2+6x-8-\dfrac{96}{x-6}\right) \div \left(3x+12-\dfrac{2x^2}{x-6}\right)$; $\dfrac{x^{a+b} \cdot x^{b+c} \cdot x^{c+a}}{x^{2c} \cdot x^{2a} \cdot x^{2b}}$.

12. $\dfrac{x^2+xy+y^2}{x^2-xy+y^2} \times \dfrac{x^3+y^3}{x^3-y^3} \times \left(\dfrac{x+y}{x-y}\right)^2$; $\left(\dfrac{a+b}{c-d}\right)^3 \times \left(\dfrac{1}{a+b}\right)^2 \div \left(\dfrac{a+b}{c-d}\right)^4$.

13. $\dfrac{(x-a)\{x^2+(2+3a)x+6a\}}{(x-c)(x^2+7ax+6a^2)} \div \dfrac{x^2+(2-c)x-2c}{x^2-a^2}$;

$(35a^{5m})^x \div (7a^3)^{mx}$.

14. $\left(\dfrac{a^9 \cdot b^{28} \cdot c^{47}}{d^{10} \cdot e^{29}}\right)^{17} \div \left(\dfrac{a^8 \cdot b^{25} \cdot c^{42}}{d^9 \cdot e^{26}}\right)^{19}$; $\left(y+\dfrac{xy}{y-x}\right)^2\left(y-\dfrac{xy}{x+y}\right) \times \dfrac{y^2-x^2}{y^2+x^2}$.

15. $\left(\dfrac{a^2-ax+x^2}{a-x} - \dfrac{a^2+ax+x^2}{a+x}\right) \times \dfrac{a^2-x^2}{x^3}$;

2,

$\left(\dfrac{a-b}{b-a}\right)^4 + \dfrac{(a-2b+3c)^5}{(2b-3c-a)^6} \div \left(\dfrac{x^2-xy}{x^2+xy} \times \dfrac{x^2y-y^2}{x^2-xy}\right)$.

16. $\dfrac{x}{2}\left(\dfrac{1}{x-y} - \dfrac{1}{x+y}\right) \times \dfrac{x^2-y^2}{x^2y+xy^2} \div \dfrac{1}{x+y}$;

1 ;

$1 \div \left(\dfrac{2ab}{3cd}\right)^3 \times \left(\dfrac{5ab}{4cd}\right)^2 \times \left(\dfrac{2cd}{5ab}\right)^4$.

17. $\dfrac{a^2-(b+c)^2}{ab+bc+ca} \times \dfrac{abc}{a^3+b^3+c^3-3abc} \div \dfrac{a^2-b^2+c^2-2ac}{a^2+b^2+c^2-ab+bc-ca}$.

18. $\left(\dfrac{x^4-a^4}{(x-a)^2} \div \dfrac{x^2+ax}{x-a}\right) \times \dfrac{x^5-a^2x^3}{x^3+a^3} \div \left(\dfrac{x}{a} - \dfrac{a}{x}\right)$;

$\left(\dfrac{1}{p} + \dfrac{1}{q}\right)\left(\dfrac{1}{p} - \dfrac{1}{q}\right) \div \dfrac{p^6-q^6}{p^6q^6}$.

132. Complex Fractions.—A fraction which has *either* term or *both* terms fractional, is called a **Complex Fraction.** Thus,

$\dfrac{x}{\frac{a}{b}}$, $\dfrac{\frac{a}{b}}{x}$, $\dfrac{\frac{x}{a}}{\frac{x}{b}}$, are complex fractions. Such fractions merely denote *division*, and are simplified by the rules for division. Thus,

$\dfrac{\frac{a}{b}}{\frac{c}{d}}$, means that $\frac{a}{b}$ is to be divided by $\frac{c}{d}$, and the result is, therefore, $\dfrac{ad}{bc}$. This result may be obtained by

(1) *Multiplying the two extreme terms* a *and* d *for a numerator, and the two middle terms* b *and* c *for a denominator.*

(2) Or by multiplying both *terms* $\left(\frac{a}{b} \text{ and } \frac{c}{d}\right)$ of the complex fraction *by the L. C. M.* (bd) *of the two denominators* b *and* d. It is evident that *any factor may be cancelled, which is common to* a *and* c, *or to* b *and* d; *i. e.*, to the two *numerators* or the two *denominators.*

Ex. 1. $\dfrac{\frac{a}{a+b}}{1-\frac{a}{a+b}}$. Reducing the denominator to a ~~mixed number~~, we have

$$\frac{\frac{a}{a+b}}{\frac{a+b-a}{a+b}} = \frac{\frac{a}{a+b}}{\frac{b}{a+b}} = \frac{a\,(a+b)}{b\,(a+b)} = \frac{a}{b}.$$

Ex. 2. $\dfrac{\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}=\dfrac{\dfrac{(x+y)^2+(x-y)^2}{x^2-y^2}}{\dfrac{(x+y)^2-(x-y)^2}{x^2-y^2}}$; and since the denominators cancel, this

$$=\frac{(x+y)^2+(x-y)^2}{(x+y)^2-(x-y)^2}=\frac{2x^2+2y^2}{4xy}=\frac{x^2+y^2}{2xy}.$$

Ex. 3. Simplify $\dfrac{1-\frac{1}{2}\left\{1-\frac{1}{3}(1-x)\right\}}{1-\frac{1}{3}\left\{1-\frac{1}{2}(1-x)\right\}}$. The L. C. M. of *all the denominators* in this case is 6, hence multiplying both terms by 6, we have,

$$\frac{6-3+1-x}{6-2+1-x}=\frac{4-x}{5-x}.$$

Ex. 4. $$\frac{\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}}{\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}}=\frac{\dfrac{(x^2+y^2)^2-(x^2-y^2)^2}{(x^2-y^2)(x^2+y^2)}}{\dfrac{(x+y)^2-(x-y)^2}{x^2-y^2}}=\frac{\dfrac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}}{\dfrac{4xy}{x^2-y^2}}$$

$$\therefore\quad \frac{4x^2y^2}{(x^2-y^2)(x^2+y^2)}\times\frac{x^2-y^2}{4xy}=\frac{xy}{x^2+y^2}.$$

Here we, in the first place, simplify the numerators and denominators *separately*, and then perform the required division.

Ex. 5. $\dfrac{9x^2-64}{x-1-\dfrac{1}{1-\dfrac{x}{x+4}}}$. Take the denominator first; by multiplying the terms of the complex fraction in the divisor by $x+4$,

$$\text{the denominator}=x-1-\frac{x+4}{x+4-x},$$

$$=\frac{4x-4-x-4}{4}=\frac{3x-8}{4}.$$

The given fraction $\therefore \quad = (9x^2 - 64) \div \dfrac{3x-8}{4}$

$$= \frac{4\,(9x^2-64)}{3x-8} = 4\,(3x+8).$$

Ex. 6. $\dfrac{x - \dfrac{x-y}{1+xy}}{1 + \dfrac{x\,(x-y)}{1+xy}}$. Here multiply both terms by $1 + xy$, and the expression

$$= \frac{x\,(1+xy)-(x-y)}{1+xy+x\,(x-y)} = \frac{x^2y+y}{1+x^2} = \frac{y\,(x^2+1)}{1+x^2} = y.$$

EXERCISE LXI.

(*a*)

Reduce to their simplest form :—

1. $\dfrac{a-\frac{3}{4}}{4}$; $\dfrac{2a+\frac{1}{4}}{7}$; $\dfrac{\frac{2}{3}x-2}{4}$; $\dfrac{\frac{a^2}{b}-c}{b}$; $\dfrac{\frac{a^2}{b^2}}{b^2}$; $\dfrac{b^2}{b}$.

$\dfrac{4a-3}{16}$ $\dfrac{8a+1}{28}$ $\dfrac{x-3}{6}$ $\dfrac{a^2-bc}{b^2}$; $\dfrac{a^2}{b^4}$ a ab

2. $\dfrac{\frac{1}{2}(a-b)}{\frac{3}{4}}$; $\dfrac{4+\frac{1}{4}x}{4-\frac{1}{4}x}$; $\dfrac{x-3\frac{1}{3}}{2x+\frac{5}{6}}$; $\dfrac{x-2\frac{1}{2}}{\frac{1}{2}x-5}$; $\dfrac{x-\frac{2}{3}}{x+\frac{3}{4}}$.

$\dfrac{2(a-b)}{3}$ $\dfrac{16+x}{16-x}$ $\dfrac{6x-20}{12x+5}$ $\dfrac{2x-5}{x-10}$ $\dfrac{12x-8}{12x+9}$

3. $\dfrac{1-\frac{a}{b}}{a-b}$; $\dfrac{1+\frac{b}{a}}{1-\frac{b}{a}}$; $\dfrac{\frac{1}{3}x-\frac{1}{4}x}{\frac{1}{2}x-\frac{5}{12}x}$; $\dfrac{\frac{x}{a}+\frac{x}{b}}{\frac{x}{a}-\frac{x}{b}}$; $\dfrac{1-\frac{1}{1+x}}{1+\frac{1}{1+x}}$.

$-\dfrac{1}{b}$ $\dfrac{a+b}{a-b}$ 1 $\dfrac{b+a}{b-a}$ $\dfrac{x}{2+x}$

4. $\dfrac{a-\frac{ab}{a+b}}{b-\frac{ab}{a+b}}$; $\dfrac{2+\frac{3a}{4b}}{a+\frac{2b}{3}}$; $\dfrac{x-\frac{1}{x}}{x+\frac{1}{x}}$; $\dfrac{x+\frac{xy}{x-y}}{x+y+\frac{y^2}{x-y}}$.

$\dfrac{a^2}{b^2}$ $\dfrac{24b+9a}{12a+8b}$ $\dfrac{x^2-1}{x^2+1}$ 1

5. $\dfrac{a-\frac{1}{1-x}}{a+\frac{1}{1-x}}$; $\dfrac{\frac{1}{1+a}}{1-\frac{1}{1+a}}$; $\dfrac{a+x-\frac{a^2+x^2}{a+x}}{a+x-\frac{2ax}{a+x}}$; $\dfrac{x-5-\frac{x^2}{x+7}}{x+2-\frac{9x+14}{x+7}}$.

$\dfrac{a-ax-1}{a-ax+1}$ $\dfrac{1}{a}$ $\dfrac{2ax}{a^2+x^2}$ $\dfrac{2x-3}{x^2}$

6. $$\frac{\frac{1}{x}-\frac{2}{x^2}-\frac{3}{x^4}}{\frac{9}{x}-x};\quad \frac{2}{1-x^2}\div\left(\frac{1}{1-x}-\frac{1}{1+x}\right);\quad \frac{\frac{a}{x^2}+\frac{x}{a^2}}{\frac{1}{a^2}-\frac{1}{ax}+\frac{1}{x^2}}.$$

$\frac{x^3-2x^2-3}{x^2(3-x)(3+x)}$; $\frac{1}{x}$; $a+x$

7. $$\frac{a+x-\frac{a^2}{a^2-ax+x^2}}{a+x-\frac{x^2}{a^2-ax+x^2}};\quad \frac{x-4-\frac{x^2-7x}{x-3}}{x+\frac{2}{x-3}};\quad \frac{\frac{1}{a+x}+\frac{1}{a-x}}{\frac{1}{a+x}-\frac{1}{a-x}}.$$ $-\frac{a}{x}$

$(a^3+x^3-a^2)\div(a^3+x^3-x^2)$; $\frac{12}{(x-1)(x-2)}$

8. $$\frac{\frac{a+b}{b}-\frac{4a}{a+b}}{\frac{a}{b}-\frac{a}{a+b}};\quad \frac{\frac{4}{a-1}+\frac{a-1}{a}}{\frac{1}{a-1}-\frac{1}{a}};\quad \frac{a^2+x^2+\frac{x^4}{a^2}}{1+\frac{x}{a}+\frac{x^2}{a^2}}.$$

$(a-b)^2\div a^2$; $(a+1)^2$; x^2-ax+x^2

9. $$\frac{1-\left(\frac{a-b}{a+b}\right)^2}{1+\left(\frac{a-b}{a+b}\right)^2};\quad \frac{\left(\frac{a+b}{a-b}\right)^2-\left(\frac{a-b}{a+b}\right)^2}{\left(\frac{a+b}{a-b}\right)^2+\left(\frac{a-b}{a+b}\right)^2};\quad \frac{\left(\frac{a+b}{a-b}\right)^3+\left(\frac{a-b}{a+b}\right)^3}{\left(\frac{a+b}{a-b}\right)^3-\left(\frac{a-b}{a+b}\right)^3}.$$

$2ab\div(a^2+b^2)$; $4ab(a^2+b^2)\div(a^4+6a^2b^2+b^4)$; $\frac{(a^2+b^2)(a^4+14a^2}{2ab(3a^4+10a}$

10. $$\frac{1-\frac{1}{2}\left\{1-\frac{1}{3}\left(1-a+b\right)\right\}}{1-\frac{1}{3}\left\{1-\frac{1}{2}\left(1-a+b\right)\right\}};\quad \frac{\frac{1}{2}\left(2a+1\frac{1}{3}\right)-\frac{1}{3}\left(2-3a\right)}{1\frac{3}{4}-\frac{1}{3}\left(2x+4\frac{1}{4}\right)}.$$

$(4-a+b)\div 5-a+b$ $6a\div(1-2x)$

11. $$\left(\frac{a+b}{a}+\frac{a-b}{b}\right)\div\left(\frac{1}{a}-\frac{1}{b}\right);\quad \frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}.$$

$\frac{a^2+b^2}{b-a}$; [illegible]

(*b*)

Reduce the following to their simplest form :—

1. $$\frac{\frac{a+b}{a-b}-\frac{a-b}{a+b}}{a-\frac{a^2+b^2}{(a+b)^2}};\quad \frac{\frac{1}{3x-2}-\frac{1}{3x+2}}{\frac{1}{a-\frac{4}{x^2}}};\quad 1+\frac{x}{1+x+\frac{2x^2}{1-x}}.$$ $\frac{1+x}{1+x^2}$

[illegible] ; $\frac{[illegible]}{x^2(3x-2)(3x+2)}$

2. $$\frac{\dfrac{x^2+y^2+z^2}{x^2+y^2-z^2}}{x^4-y^4-z^4-2y^2z^2};\quad \frac{\dfrac{x}{3}-\dfrac{x}{4}}{\dfrac{x}{3}+\dfrac{x}{4}}-\frac{\dfrac{x}{4}-\dfrac{x}{5}}{\dfrac{x}{4}+\dfrac{x}{5}};\quad \frac{a-\dfrac{2a}{a+3}}{a+\dfrac{2a}{a-3}}\times\frac{1}{a+1}.$$

$\frac{1}{(x^2-y^2-z^2)(x^2 \ldots)}$, $\frac{2}{63}$; $\frac{a-3}{(a-1)(a+3)}$

3. $$\frac{1-\dfrac{2ab}{(a+b)^2}}{1+\dfrac{2ab}{(a-b)^2}}\div\frac{1-\dfrac{a}{b}}{1+\dfrac{b}{a}};\quad \frac{(ax+by)^2+(ay-bx)^2}{\left(\dfrac{a}{x}+\dfrac{b}{y}\right)^2+\left(\dfrac{a}{y}-\dfrac{b}{x}\right)^2}.$$

1 $\frac{b^2-ab}{a^2+ab}$ x^2y^2

4. $$\frac{\left(\dfrac{x}{x-1}-\dfrac{1}{x+1}\right)\left(\dfrac{x^3-1}{x^6+1}\right)}{\dfrac{x^4+x^2+1}{(x-1)^2(x+1)^2+x^2}};\quad \frac{1}{x+\dfrac{1}{x+\dfrac{1}{x+\dfrac{1}{x}}}}.$$

$\frac{1}{x^3+1}$ $\frac{x(x^2+2)}{x^4+3x^2+1}$

5. $$\frac{\dfrac{1}{m^4+m^2n^2+n^4}}{\dfrac{m+n}{m^3+n^3}}\div\frac{m-n}{m^3-n^3};\quad \frac{\dfrac{1}{a}}{\dfrac{1}{a}-\dfrac{1}{b}}\div\frac{(a+b)^2-4ab}{\dfrac{1}{a^2}+\dfrac{1}{b^2}-\dfrac{2}{ab}}.$$

/ $a^2b/(a-b)$

*

6. $$\frac{\dfrac{1}{x}-\dfrac{x+a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a+x}{a^2+x^2}}+\frac{\dfrac{1}{x}-\dfrac{x-a}{x^2+a^2}}{\dfrac{1}{a}-\dfrac{a-x}{a^2+x^2}};\quad \frac{1}{4x+\dfrac{4x}{1+\dfrac{2(x+y)}{6-x}}}.$$

0 $\frac{x+2\ \ +6}{6x(y+x)}$

7. $$\frac{\dfrac{x^2+y^2}{y}+x}{\dfrac{1}{y}+\dfrac{1}{x}}\times\frac{x^2-y^2}{y^3-x^3};\quad \frac{1}{1+\dfrac{1}{1+\dfrac{x}{1+x}}}.$$

$-x$ $\frac{1+2x}{2+3x}$

8. $$\frac{a^2-x^2}{a^2+ax+x^2}\div\frac{\left(1-\dfrac{x}{a}\right)^3\left(1+\dfrac{x}{a}\right)^3}{a^3-x^3};$$

$\frac{a^6}{(a+x)(a^2-x^2)}$

$$\left(\frac{1-x^2}{1-x^3}+\frac{1-x}{1-x+x^2}\right)\div\left(\frac{1+x}{1+x+x^2}-\frac{1-x^2}{1+x^3}\right).$$

$\frac{1}{x}$

9. $\dfrac{\left(\dfrac{a+b}{a-b}\right)^2 - 2 + \left(\dfrac{a-b}{a+b}\right)^2}{\left(\dfrac{a+b}{a-b}\right)^2 + 2 + \left(\dfrac{a-b}{a+b}\right)^2}$; $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}-1}{\dfrac{x^3}{y^3}-1} \times \dfrac{\dfrac{x}{y}+\dfrac{y}{x}+1}{\dfrac{x^3}{y^3}+1} \times \left(\dfrac{x^4}{y^4}-\dfrac{x^2}{y^2}\right)$.

$\dfrac{4a^2b^2}{(a^2+b^2)^2}$ 　 /

(c)

Simplify the following :—

1. $\dfrac{1-a^2}{(1+ax)^2-(a+x)^2} \div \dfrac{1}{2}\left(\dfrac{1}{1-x}+\dfrac{1}{1+x}\right)$; $\dfrac{a^2}{\dfrac{a^2+b^2}{a+b}-b} + \dfrac{b^2}{\dfrac{a^2+b^2}{a+b}-a}$.

/ , $a+b$

2. $21\left(\dfrac{2x+4y}{3}-\dfrac{4x-y}{7}\right)-24\left[\dfrac{2x-y}{12}-\left\{\dfrac{5x-7y}{8}-\dfrac{1}{6}\left(\dfrac{x}{2}-\dfrac{5x-y}{4}\right)\right\}\right]$. $16x$

3. $12\left(\dfrac{2x-y}{3}-\dfrac{x-3y}{4}\right)+16\left(\dfrac{x+2y}{8}-\dfrac{x-\frac{1}{2}(x+y)}{4}\right)$. $5x+11y$

4. $\dfrac{x+y+\dfrac{y^2}{x}}{x-y+\dfrac{y^2}{x}} \times \dfrac{x+\dfrac{y^3}{x^2}}{x-\dfrac{y^3}{x^2}} \div \left(\dfrac{x+y}{x-y}\right)^2$; $\left\{\dfrac{1}{3}+\dfrac{2x}{3(1-x)}\right\} \times \left\{\dfrac{3}{4}-\dfrac{3x}{2(1+x)}\right\}$.

$\dfrac{3-2x}{4+4x}$ 　 $\dfrac{1}{4}$

5. $\dfrac{1}{\left(1-\dfrac{c}{a}\right)\left(1-\dfrac{b}{a}\right)} + \dfrac{1}{\left(1-\dfrac{a}{b}\right)\left(1-\dfrac{c}{b}\right)} + \dfrac{1}{\left(1-\dfrac{b}{c}\right)\left(1-\dfrac{a}{c}\right)}$. 　 /

6. $\left(\dfrac{1+x}{1-x}+\dfrac{4x}{1+x^2}+\dfrac{8x}{1+x^4}-\dfrac{1-x}{1+x}\right) \div \left(\dfrac{1+x^2}{1-x^2}+\dfrac{4x^2}{1+x^4}-\dfrac{1-x^2}{1+x^2}\right)$. 　 $\dfrac{2}{x}$

7. $\dfrac{1}{a+b} \div \left\{\dfrac{b}{2}\left(\dfrac{1}{a+b}+\dfrac{1}{a-b}\right) \times \dfrac{a^2-b^2}{a^2b-b^2a}\right\}$. 　 $\dfrac{a-b}{a+b}$

8. $\left(\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right) \div \left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right)$. 　 $\dfrac{a}{2x^2}$

9. $\left(\dfrac{b+\dfrac{a-b}{1+ab}}{1-\dfrac{b(a-b)}{1+ab}} - \dfrac{a-\dfrac{a-b}{1-ab}}{1-\dfrac{a(a-b)}{1-ab}}\right) \div \left(\dfrac{a}{b}-\dfrac{b}{a}\right)$. 　 $\dfrac{ab}{a+b}$

10. Find the value of $\dfrac{x+2a}{x-2a} + \dfrac{x+2b}{x-2b}$, when $x = \dfrac{4ab}{a+b}$.

11. Find the value of $\dfrac{a + \sqrt{a^2+b^2}}{a^3 - 2b(a^2-b^2)}$, when $a = -4$, and $b = -3$.

12. Find the value of $ax + by$, when $x = \dfrac{cq - br}{aq - bp}$, and $y = \dfrac{ar - cp}{aq - bp}$.

13. Show that the value of $\dfrac{a^n}{2na^n - 2nx} + \dfrac{b^n}{2nb^n - 2nx}$, is $\dfrac{1}{n}$, when $x = \frac{1}{2}(a^n + b^n)$.

14. Prove that the value of $\left(\dfrac{x}{x-1}\right)^2 + \left(\dfrac{x}{x+1}\right)^2$ is $n(n-1)$, when $x = \sqrt{\left(\dfrac{n-1}{n+1}\right)}$.

15. Show that $\left(\dfrac{x+a}{x+b}\right)^3 - \dfrac{x+2a+b}{x-a-2b}$ is equal to zero, when $x = \frac{1}{2}(b-a)$.

16. If $2s = a + b + c$, prove that

$$\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} - \frac{1}{s} = \frac{abc}{s(s-a)(s-b)(s-c)}.$$

17. Find the value of $\dfrac{x+y-1}{x-y+1}$, when $x = \dfrac{a+1}{ab+1}$, and $y = \dfrac{ab+a}{ab+1}$.

18. Show that

$$\frac{a^2\left(\frac{1}{c^2} - \frac{1}{b^2}\right) + b^2\left(\frac{1}{a^2} - \frac{1}{c^2}\right) + c^2\left(\frac{1}{b^2} - \frac{1}{a^2}\right)}{\frac{1}{bc}\left(\frac{1}{c} - \frac{1}{b}\right) + \frac{1}{ca}\left(\frac{1}{a} - \frac{1}{c}\right) + \frac{1}{ab}\left(\frac{1}{b} - \frac{1}{a}\right)}$$

is equal to $(a+b)(b+c)(c+a)$.

19. If $s = a + b + c + \ldots.$ to n terms, show that

$$\frac{s-a}{a} + \frac{s-b}{b} + \frac{s-c}{c} + \ldots \text{ to } n \text{ terms} = s\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \ldots\right) - n.$$

20. If $\dfrac{a+b}{1-ab} = \dfrac{c+d}{cd-1}$, show that $\dfrac{a+b+c+d}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}} = abcd.$

133. Ratios.—The fraction $\frac{a}{b}$ expresses a relation between the two quantities a and b, which is called their *ratio*. When *two* fractions, or ratios, are equal, they constitute a *proportion*, thus,

$$\frac{a}{b} = \frac{c}{d} \quad \text{means} \quad a : b :: c : d.$$

The student should carefully attend to the following results deduced from the equality of two fractions, or ratios.

$$\text{If} \qquad \frac{a}{b} = \frac{c}{d}, \quad \text{then} \quad ad = bc.$$

Divide both sides of this identity by ac, and we have

$$\frac{b}{a} = \frac{d}{c}. \tag{1}$$

Divide both sides of the same identity by cd, and we have

$$\frac{a}{c} = \frac{b}{d}. \tag{2}$$

Divide both sides of the identity by ab, and we have

$$\frac{d}{b} = \frac{c}{a}. \tag{3}$$

Add 1 to both sides of $\frac{a}{b} = \frac{c}{d}$, and we get

$$\frac{a}{b} + 1 = \frac{c}{d} + 1;$$

$$i.e. \quad \frac{a+b}{b} = \frac{c+d}{d}. \tag{4}$$

Similarly, by subtracting 1 from both sides,

$$\frac{a-b}{b} = \frac{c-d}{d}. \tag{5}$$

Divide (5) by (4), and we have

$$\frac{a-b}{a+b} = \frac{c-d}{c+d}. \tag{6}$$

Also: $$\frac{a}{b} = \frac{c}{d} = \frac{ma}{mb} = \frac{nc}{nd} = x, \text{ (suppose)},$$

$$\therefore \quad ma = mbx, \quad nc = ndx;$$

$$\therefore \quad \frac{ma + nc}{mb + nd} = x = \frac{a}{b} = \frac{c}{d};$$

i. e. = each fraction. (7)

By putting $m = 1$, $n = \pm 1$, we get the useful result

$$\frac{a}{b} = \frac{c}{d} = \frac{a - c}{b - d} = \frac{a + c}{b + d}. \tag{8}$$

Similarly, if $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f},$$

then each fraction $$= \frac{ma}{mb} = \frac{nc}{nd} = \frac{pe}{pf}.$$

Suppose each of these fractions, or ratios, $= x$,

$$\therefore \quad ma = mbx, \quad nc = ndx, \quad pe = pfx;$$

$$\therefore \quad \frac{ma + nc + pe}{mb + nd + pf} = x = \frac{a}{b} = \frac{c}{d} = \frac{e}{f};$$

i. e. = each fraction. (9)

Generally, if $\frac{a}{b} = \frac{c}{d}$, *then any fraction formed with numerator and denominator* **homogeneous in** a **and** b, *will be equal to a fraction similarly formed of* c *and* d. This can be proved in any case by putting each fraction equal x, so that $a = bx$, $c = dx$; for, on making the substitutions, b will cancel from all terms of the first fraction and d from all terms of the second fraction, *leaving two fractions which are identical in terms of* x.

We shall now give a few examples of the application of the formulas proved above.

Ex. 1. If $$\frac{a}{b} = \frac{c}{d},$$

then $$\frac{5a - 7b}{5a - 8b} = \frac{5c - 7d}{5c - 8d}.$$

Divide both terms of the first fraction by b, and there results

$$\frac{5\frac{a}{b}-7}{5\frac{a}{b}-8}=\frac{5\frac{c}{d}-7}{5\frac{c}{d}-8}=\frac{5c-7d}{5c-8d},$$

by putting $\frac{c}{d}$ for its equal $\frac{a}{b}\cdot$

Ex. 2. If $$\frac{a}{b}=\frac{c}{d},$$

prove $$\frac{a^3+ab^2+b^3}{a^3-ab^2+b^3}=\frac{c^3+cd^2+d^3}{c^3-cd^2+d^3}.$$

Divide both terms of first fraction by b^3, and we have

$$\frac{\frac{a^3}{b^3}+\frac{a}{b}+1}{\frac{a^3}{b^3}-\frac{a}{b}+1}=\frac{\frac{c^3}{d^3}+\frac{c}{d}+1}{\frac{c^3}{d^3}-\frac{c}{d}+1}=\frac{c^3+cd^2+d^3}{c^3-cd^2+d^3}.$$

Ex. 3. If $\dfrac{b^2+c^2-a^2}{bc}=\dfrac{c^2+a^2-b^2}{ca}=\dfrac{a^2+b^2-c^2}{ab}$,

then each of these fractions is equal to 1. Take the first and second together, then, by (8), each fraction

$$=\frac{b^2+c^2-a^2+c^2+a^2-b^2}{bc+ca}=\frac{2c^2}{c(a+b)}=\frac{2c}{a+b}=\frac{2a}{b+c}=\frac{2b}{c+a},$$

by symmetry; $$=\frac{2a+2b+2c}{a+b+b+c+c+a}=1.$$

Ex. 4. If $$\frac{x^3+ax^2-bx+c}{x^3-ax^2+bx+c}=\frac{x^2+ax-b}{x^2-ax+b},$$

find x in terms of a and b. Multiply numerator and denominator of right side by x and apply (8), and each fraction becomes

$$=\frac{c}{c}=1.$$

$$\therefore\quad x^2+ax-b=x^2-ax+b;$$

$$\therefore\quad 2ax=2b;\quad\therefore\quad x=\frac{b}{a}\cdot$$

Ex. 5. If $$\frac{a+b}{a-b} = \frac{b+c}{2(b-c)} = \frac{c+a}{3(c-a)},$$

then $$8a + 9b + 5c = 0.$$

Put each of the given fractions $= x$, so that

$$a + b = x(a - b),\ b + c = x\cdot 2(b - c),\ c + a = x \cdot 3(c - a).$$

$$\therefore a + b + \frac{b+c}{2} + \frac{c+a}{3} = \{(a-b) + (b-c) + (c-a)\}\,x = 0.$$

Clearing of fractions $\therefore\ 8a + 9b + 5c = 0.$

Ex. 6. If

$$(a + b + c + d)(a - b - c + d) = (a - b + c - d)(a + b - c - d),$$

show that $$\frac{a}{b} = \frac{c}{d}.$$

By division we have

$$\frac{a+b+c+d}{a-b+c-d} = \frac{a+b-c-d}{a-b-c+d}.$$

Now applying (6) we get

$$\frac{2(a+c)}{2(b+d)} = \frac{2(a-c)}{2(b-d)} = \frac{(a+c)+(a-c)}{(b+d)+(b-d)} = \frac{2a}{2b} = \frac{a}{b} \quad \text{(by 8).}$$

Also $$= \frac{(a+c)-(a-c)}{(b+d)-(b-d)} = \frac{2c}{2d} = \frac{c}{d}.$$

134. The student should observe that, by actually dividing, we may obtain any number of terms of an inexact quotient, *e. g.* $\dfrac{1}{1-bx}$;

$$\begin{array}{l|l}
1 & 1 - bx \\
\underline{1 - bx} & \underline{1 + bx + b^2x^2 + \ldots} \text{ to as many terms as we please.} \\
\quad bx & \\
\quad \underline{bx - b^2x^2} & \\
\qquad b^2x^2 & \\
\qquad \underline{b^2x^2 - b^3x^3} & \\
\qquad\qquad b^3x^3 &
\end{array}$$

EXERCISE LXII.

If $\frac{a}{b} = \frac{c}{d}$, prove the relations in 1, 2, and 3 :—

✓ 1. $\frac{3a - 2b}{3a + 4b} = \frac{3c - 2d}{3c + 4d}$; $\frac{2a + 3b}{4a + 5b} = \frac{2c + 3d}{4c + 5d}$.

✓ 2. $\frac{3a}{2a - 7b} = \frac{3c}{2c - 7d}$; $\frac{a^2 - ab + b^2}{a^2 + ab + b^2} = \frac{c^2 - cd + d^2}{c^2 + cd + d^2}$.

✓ 3. $\frac{a^2 + ab}{c^2 + cd} = \frac{ab - b^2}{cd - d^2}$; $\frac{a^3 + 2a^2b}{ab^2 - 3b^3} = \frac{c^3 + 2c^2d}{cd^2 - 3d^3}$; $\frac{a^3 + 3b^3}{a^3 - 3b^3} = \frac{c^3 + 3d^3}{c^3 - 3d^3}$.

✓ 4. If $\frac{x}{y} = \frac{a^2}{b^2}$, prove that $\frac{5a^4 + 7b^4}{5x^2 + 7y^2} = \frac{7a^4 - 5b^4}{7x^2 - 5y^2}$.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, show that

✓ 5. $\frac{a}{b} = \frac{c + e}{d + f} = \frac{ma - ne}{mb - nf} = \frac{a - c - e}{b - d - f} = \frac{ma - nc - pe}{mb - nd - pf}$

$= \frac{m(a + e) - nc}{m(b + f) - nd}$.

✓ 6. And that $\frac{a^2}{b^2} = \frac{a^2 - e^2}{b^2 - f^2} = \frac{(a + c)^2}{(b + d)^2} = \frac{mc^2 - ne^2}{md^2 - nf^2} = \frac{a^2 + c^2 + e^2}{b^2 + d^2 + f^2}$

$= \frac{(a - mc + ne)^2}{(b - md + nf)^2}$.

✓ 7. If $\frac{a + b}{3(a - b)} = \frac{b + c}{4(b - c)} = \frac{c + a}{5(c - a)}$, prove that
$32a + 35b + 27c = 0$.

✓ 8. If $\frac{x}{y + z} = \frac{y}{z + x} = \frac{z}{x + y}$, prove each ratio $= \frac{1}{2}$.

✓ 9. If $\frac{a}{b + c} = \frac{b}{c + a} = \frac{c}{a + b}$, prove $a = b = c$.

10. If $\frac{a}{b} = \frac{c}{d}$, prove $\frac{(a + c)(a^2 + c^2)}{(a - c)(a^2 - c^2)} = \frac{(b + d)(b^2 + d^2)}{(b - d)(b^2 - d^2)}$.

✓ 11. If $(3a+6b+c+2d)(3a-6b-c+2d)$
$= (3a-6b+c-2d)(3a+6b-c-2d)$, prove $\frac{a}{b}=\frac{c}{d}$.

✓ 12. If $\frac{a}{b}=\frac{c}{d}$, prove $\frac{a^3-a^2b+ab^2-3b^3}{c^3-c^2d+cd^2-3d^3}=\frac{a^3+a^2b+ab^2+3b^3}{c^3+c^2d+cd^2+3d^3}$.

✗

✓ 13. Prove, also,

i. $\frac{b}{d}\times\frac{a^3+b^3}{c^3+d^3}=\left(\frac{a+b}{c+d}\right)^4$. ii. $\frac{(a^3-c^3)^2}{(b^3-d^3)^2}=\frac{(a^2-c^2)^3}{(b^2-d^2)^3}$.

14. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove

✓ i. $\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}$.

✓ ii. $\frac{x^2+y^2+z^2}{a^2+b^2+c^2}=\frac{(x+y+z)^2}{(a+b+c)^2}=\frac{x^2}{a^2}$.

✓ 15. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, prove
$(b-c)x+(c-a)y+(a-b)z=0$.

✓ 16. If $\frac{cy+bz}{b-c}=\frac{az+cx}{c-a}=\frac{bx+ay}{a-b}$, then will
$(a+b+c)(x+y+z)=ax+by+cz$.

✓ 17. If $\frac{ay-bx}{c}=\frac{bz-cy}{a}=\frac{cx-az}{b}$, then will $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

18. If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, prove

✓ i. $\frac{xyz}{abc}=\left(\frac{x+y+z}{a+b+c}\right)^3=\frac{x^3+y^3+z^3}{a^3+b^3+c^3}$;

✓ ii. $\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=\frac{x^3}{a^3}$.

✓ 19. If $\frac{a^2}{x^2-yz}=\frac{b^2}{y^2-zx}=\frac{c^2}{z^2-xy}$, prove that
$a^2x+b^2y+c^2z=(a^2+b^2+c^2)(x+y+z)$.

✓ 20. If $\frac{x}{a(y-z)}=\frac{y}{b(z-x)}=\frac{z}{c(x-y)}$, then will $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$.

21. Obtain the quotients to five terms in each of the following cases :—

1. $\dfrac{1}{1-x}$. 2. $\dfrac{1}{1-3x}$. 3. $\dfrac{1}{1+x}$. 4. $\dfrac{1}{1+3x}$. 5. $\dfrac{a^2}{x-b}$.

6. $\dfrac{ax}{a-x}$. 7. $\dfrac{a}{1-bx}$. 8. $\dfrac{a^2}{(a+x)^2}$. 9. $\dfrac{1}{1-x+x^2}$.

10. $\dfrac{1}{1-ax+bx^2}$.

22. Show that $\dfrac{a^2-(b+c)^2}{(a+b)^2-c^2}$ + anal. + anal. $= 1$.

23. Show that
$$\frac{a^4}{(a-b)(a-c)(a-d)}+\frac{b^4}{(b-c)(b-d)(b-a)}$$
$$+\frac{c^4}{(c-d)(c-a)(c-b)}+\frac{d^4}{(d-a)(d-b)(d-c)}=a+b+c+d.$$

24. Show that $\Sigma\,\dfrac{bcd}{(a-b)(a-c)(a-d)}=1$.

25. Show that $\Sigma\,\dfrac{bc(x-a)^2}{(a-b)(a-c)}=x^2$.

26. If $a+b+c=0$, show that $\Sigma\,\dfrac{a^2}{2a^2+bc}=1$.

27. If $x+y=m$ and $x-y=n$, show that $\dfrac{x^3+y^3}{x^3-y^3}=\dfrac{m(m^2+3n^2)}{n(n^2+3m^2)}$.

28. If $x+y=a$ and $x-y=b$, show that
$$\frac{x^4-2x^3y+2xy^3-y^4}{x^4-47x^2y^2+y^4}=\frac{16ab^3}{(9a^2-5b^2)(9b^2-5a^2)}.$$

29. If $xy=ab(a+b)$ and $x^2-xy+y^2=a^3+b^3$, then
$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{b}-\frac{y}{a}\right)=0.$$

30. Prove $\dfrac{b^2c^2(b-c)+c^2a^2(c-a)+a^2b^2(a-b)}{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}=1$.

31. If $x+y=2a$ and $x-y=2b$, show that
$$\frac{x^4+x^2y^2+y^4}{x^4-23x^2y^2+y^4}=\frac{(3a^2+b^2)(3b^2+a^2)}{(7a^2-3b^2)(7b^2-3a^2)}.$$

CHAPTER XII.

MISCELLANEOUS FRACTIONAL EQUATIONS.

135. We shall now consider simple equations involving fractions of greater difficulty than those found in the equations of Chapter VI. We saw that the general rule for *clearing of fractions* is :—*Multiply both sides of the equation by the L. C. M. of all the denominators that occur in the equation.* The student will find, however, that by using the artifices employed in reducing fractions (Art. 129), he may greatly simplify the work of solving fractional equations.

Ex. 1. $\frac{6x+13}{15} - \frac{3x+5}{5x-25} = \frac{2x}{5}$.

Here the L. C. M. is $15(x-5)$; multiplying both sides by this, we have

$$(x-5)(6x+13) - 3(3x+5) = 3(x-5)\cdot 2x,$$

that is, $6x^2 + 13x - 30x - 65 - 9x - 15 = 6x^2 - 30x$,

or, by cancelling terms, $13x - 65 - 9x - 15 = 0$,

that is, $13x - 9x = 80$

$$\therefore\ 4x = 80, \text{ and } x = 20.$$

The factor 5 which is found in all the denominators might have been struck out, making the several denominators 3, $x-5$, and 1, respectively, the L. C. M. of which is $3(x-5)$.

Also, we might have *first* combined the first and last fractions, and struck the factor 5 out of the denominators, getting at once

$$\frac{13}{3} - \frac{3x+5}{x-5} = 0, \text{ or } 13x - 65 - 9x - 15 = 0, \text{ or } 4x = 80, \text{ etc.}$$

Ex. 2. $\frac{10x+17}{18} - \frac{12x+2}{11x-8} = \frac{5x-4}{9}$.

The L. C. M. is $18(11x-8)$. But instead of multiplying by this, combine the first and last fractions, and we get at once

$$\frac{10x+17-10x+8}{18}-\frac{12x+2}{11x-8}=0,$$

or

$$\frac{25}{18}-\frac{12x+2}{11x-8}=0,$$

now clear of fractions,

$$\therefore\ 275x-200-216x-36=0,$$

or

$$59x=236,\text{ and } x=4.$$

Ex. 3. $\dfrac{21}{x-98}-\dfrac{71}{x-94}=\dfrac{21}{x+44}-\dfrac{71}{x-52}.$

Here first combine the pairs that have like numerators, separating the numerator as a factor,

$$21\left(\frac{1}{x-98}-\frac{1}{x+44}\right)=71\left(\frac{1}{x-94}-\frac{1}{x-52}\right).$$

$$\therefore\ 21\left\{\frac{x+44-x+98}{(x-98)(x+44)}\right\}=71\left\{\frac{x-52-x+94}{(x-94)(x-52)}\right\},$$

that is,

$$\frac{21\times 142}{(x-98)(x+44)}=\frac{71\times 42}{(x-94)(x-52)},$$

where the numerators are equal and, therefore, the denominators also, that is

$$(x-98)(x+44)=(x-94)(x-52), \tag{1}$$

or

$$x^2-54x-98\times 44=x^2-146x+94\times 52$$
$$\therefore\ 92x=98\times 44+52\times 94,$$

or

$$23x=98\times 11+13\times 94=2300$$
$$\therefore\ x=100.$$

We might have got from 1, by division,

$$\frac{x-98}{x-94}=\frac{x-52}{x+44}=\therefore\text{ (Art. 133, 8) }\frac{x-98-(x-52)}{(x-94)-(x+44)}=\frac{-\ 46}{-138}=\frac{1}{3}.$$

$$i.\ e.\ \frac{x-52}{x+44}=\frac{1}{3}\ \therefore\ 3x-156=x+44,\text{ and } 2x=200.$$

Ex. 4. $\frac{x}{x-2}+\frac{x-9}{x-7}=\frac{x+1}{x-1}+\frac{x-8}{x-6}.$

Here complete the divisions, thus,

$$1+\frac{2}{x-2}+1-\frac{2}{x-7}=1+\frac{2}{x-1}+1-\frac{2}{x-6};$$

or $$\frac{2}{x-2}-\frac{2}{x-7}=\frac{2}{x-1}-\frac{2}{x-6},$$

where 2 divides out. Combine,

$$\therefore\ \frac{x-7-(x-2)}{(x-2)(x-7)}=\frac{x-6-(x-1)}{(x-1)(x-6)};$$

or $$\frac{-5}{(x-2)(x-7)}=\frac{-5}{(x-1)(x-6)},$$

where the numerators are equal, and therefore the denominators are equal, or

$$(x-2)(x-7)=(x-1)(x-6);$$

or $$x^2-9x+14=x^2-7x+6;$$

$$\therefore\ -9x+7x=-14+6,$$

and $$x=4.$$

Ex. 5. $\frac{5x-8}{x-2}+\frac{6x-44}{x-7}-\frac{10x-8}{x-1}=\frac{x-8}{x-6}.$

Here complete the divisions, thus

$$5+\frac{2}{x-2}+6-\frac{2}{x-7}-10-\frac{2}{x-1}=1-\frac{2}{x-6}.$$

Simplify, and divide by 2,

$$\therefore\ \frac{1}{x-2}+\frac{1}{x-6}=\frac{1}{x-7}+\frac{1}{x-1};$$

or $$x^2-8x+12=x^2-8x+7;$$

$$\therefore\ (8-8)x+12=7,\ i.e.\ 0\cdot x=-5.$$

which can be true only if x is *infinitely great.* If we combine the fractions differently we get

$$\frac{1}{x-2}-\frac{1}{x-7}=\frac{1}{x-1}-\frac{1}{x-6};$$

or $$x^2-9x+14=x^2-7x+6;$$

$$\therefore\quad -2x=-8,$$

and $$x=4.$$

N. B.—The student will learn *from the above examples that labour may be saved by combining fractions in pairs, changing them to mixed numbers, etc.*

EXERCISE LXIII.

(*a*.)

Solve :—

1. $\frac{6}{x-1}=\frac{5}{x-2}$. 7

2. $\frac{5}{3x-2}=\frac{19}{7x-1}$. $1\frac{1}{2}$

3. $\frac{x}{x-1}+\frac{x-2}{x-3}=2$. 2

4. $\frac{x-1}{x-2}+\frac{x-3}{x-4}=2$. 3

5. $\frac{x-4}{x-5}=2-\frac{x-2}{x-3}$. 4

6. $\frac{x}{x-3}-1=\frac{14}{x-7}$. $\frac{21}{11}$

7. $\frac{x}{x-1}+\frac{x+1}{x}=2$. $\frac{1}{2}$

8. $x-1-\frac{x^2+3}{x+2}=0$. 5

9. $\frac{1}{x}-\frac{1}{7x}=\frac{2}{x+8}$. 6

10. $\frac{x+a}{x-b}=a-\frac{x}{x-b}$. $\frac{a(1+b)}{a-2}$

11. $\frac{a-b}{x-b}=\frac{a+b}{x+2b}$. $\frac{3a-b}{2}$

12. $\frac{1}{a+bx}=\frac{1}{b+ax}$. 1

13. $\frac{x}{a}+\frac{x}{b-a}=\frac{a}{a+b}$. $\frac{a^2(b-a)}{b(a+b)}$

14. $\frac{3a+x}{x}=5+\frac{6}{x}$. $\frac{3a-6}{4}$

15. $\frac{1}{x-a}+\frac{1}{x-b}=\frac{2}{x}$. $\frac{2ab}{a+b}$

16. $\frac{x}{a+c}-\frac{x+d}{a+b+c}=0$. $\frac{d(a+c)}{b}$

17. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a^2-b^2}{x^2-b^2}$. a

a(a+2b)

18. $\frac{6}{x-1}-\frac{4x}{x^2-1}=\frac{3}{x+1}$.

19. $\frac{x^2+1}{4x^2-1}+\frac{1}{4}-\frac{x}{2x+1}=0$. $-\frac{3}{4}$

20. $\frac{4}{x}+\frac{7}{x+1}=\frac{37}{x^2+x}$. 3

21. $\frac{\frac{1}{5}x+8}{\frac{1}{15}x+1}-\frac{\frac{1}{15}x+38}{\frac{1}{30}x+12}=1$. 60

22. $\frac{4}{x+2}+\frac{7}{x+3}=\frac{37}{x^2+5x+6}$.

23. $\frac{7}{x-1}=\frac{6x+1}{x+1}-\frac{3+6x^2}{x^2-1}$. $-\frac{11}{12}$

24. $\frac{6}{x-2}+\frac{4(x-1)}{x^2-2x}=\frac{3}{x}$. 2

25. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}$.

ab ÷ (a+b−c)

26. $\frac{x-a}{a-b}=\frac{x+a}{a+b}+\frac{2ax}{a^2-b^2}$.

a² ÷ (b−a)

27. $\frac{6x+7}{9x+6}=\frac{1}{12}+\frac{5x-5}{12x+8}$. $-6\frac{5}{6}$

28. $\frac{2x+5}{5x+3}-\frac{2x+1}{5x+2}=0$. $-\frac{7}{18}$

$1+\frac{2x+3}{2x^2-7x+3}=1+\frac{2x+3}{x^2-9x+2}$

29. $\frac{2x^2-5x+6}{2x^2-7x+3}=\frac{x^2-7x+5}{x^2-9x+2}$.

$-\frac{3}{2}$

30. $\frac{12x+10a}{3x+a}+\frac{28x+117a}{2x+9a}=18$.

3a

31. $\frac{ax+b}{x-m}+\frac{cx+d}{x-n}=a+c$.

(bn+dm+cmn+amn ÷ (b+d+am−cn [illegible]

32. $\frac{132x+1}{3x+1}+\frac{8x+5}{x-1}=52$.

14

33. $\frac{m}{x+a}+\frac{n}{x-b}=\frac{m+n}{x-c}$.

(acn−abn [illegible] bm−bcm ÷ (bn−cn−am− [illegible]

34. $\frac{2x+a}{3(x-a)}+\frac{3x-a}{2(x+a)}=\frac{1}{3}$. $2\frac{1}{6}$

3a

35. $\frac{x-2}{2x+1}+\frac{x-1}{3(x-3)}=\frac{5}{6}$. 7

36. $\frac{5x^2+x-3}{5x-4}=\frac{7x^2-3x-9}{7x-10}$. 3

37. $\frac{x+1}{4(x+2)}+\frac{x+4}{5x+13}=\frac{9}{20}$. 3

38. $\frac{1+x}{1-x}-\frac{2+3x}{2-3x}=1+\frac{1+3x}{1-3x}$. $\frac{1}{2}$

39. $\frac{6}{x-3}-\frac{4(x-2)}{x^2-4x+3}=\frac{3}{x-1}$. 11

40. $\frac{x+3}{\frac{1}{2}x+1}=\frac{2x+5}{2x+4}+\frac{\frac{3}{2}x+3}{\frac{3}{2}x+1}$. −6

10

(*b.*)

Solve :—

1. $\frac{25-\frac{1}{3}x}{x+1}+\frac{16x+4\frac{1}{5}}{3x+2}=5+\frac{23}{x+1}$.

$+3\frac{3}{8}$

2. $\dfrac{7x+1}{x-1} = \dfrac{35}{9}\left(\dfrac{x+4}{x+2}\right) + \dfrac{28}{9}.$ -127

3. $\dfrac{mx+a+b}{nx-c-d} = \dfrac{mx+a+c}{nx-b-d}.$ $\dfrac{a+b+c-2d}{n-m}$

4. $\dfrac{1}{2}\left(\dfrac{2}{3}x+4\right) - \dfrac{7\frac{1}{2}-x}{3} = \dfrac{x}{2}\left(\dfrac{6}{x}-1\right).$ 3

5. $\dfrac{1}{ab-ax} + \dfrac{1}{bc-bx} - \dfrac{1}{ac-ax} = 0.$ $\dfrac{b(a-b+c)}{a}$

6. $\dfrac{a}{bx} + \dfrac{b}{ax} = a^2 + b^2.$ $\dfrac{1}{ab}$

7. $\dfrac{x}{2} - \dfrac{\frac{1}{3}(2x-3) - \frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)} = \dfrac{3}{2} \times \dfrac{x^2+2}{3x-2}.$ 0 or +4

8. $\dfrac{4x+3}{2x+1} - \dfrac{3}{3x+2} - \dfrac{4x-9}{2x+1} = 0.$ $-\dfrac{7}{10}$

9. $\dfrac{a(x^2+m^2)}{mx} = ac + \dfrac{ax}{m}.$ $\dfrac{m}{c}$

10. $\dfrac{1}{3}(20\frac{5}{8} - \frac{4}{3}x) + \frac{1}{3}x = 6\frac{5}{12} - \dfrac{\frac{8}{3}x-15}{24} - \dfrac{x-13\frac{1}{3}}{15-\frac{1}{3}x}.$ 15

← 11. $\dfrac{x}{x-1} \overset{-}{\cancel{+}} \dfrac{x-1}{x-2} = \dfrac{x-3}{x-4} - \dfrac{x-4}{x-5}.$ 3

12. $\dfrac{2x-3}{.3x-.4} = \dfrac{.4x-.6}{.06x-.07}.$ $1\frac{1}{2}$

13. $\dfrac{3+x}{1+x} + \dfrac{x+5}{x+7} - 1 = \dfrac{x^2-1}{7+8x+x^2}.$ $-2\frac{1}{2}$

14. $\dfrac{x+10}{x+8} + \dfrac{x+1}{x+3} = \dfrac{x+11}{x+9} + \dfrac{x+2}{x+4}.$ -6

15. $\dfrac{x-4}{x-5} - \dfrac{x-5}{x-6} = \dfrac{x-7}{x-8} - \dfrac{x-8}{x-9}.$ 7

16. $\dfrac{x+m}{x^2+mx+m^2} - \dfrac{x-m}{x^2-mx+m^2} = \dfrac{m^4}{x(x^4+m^2x^2+m^4)}.$

17. $\frac{x^2-5}{x^2-6}+\frac{x^2-11}{x^2-12}=\frac{x^2-7}{x^2-8}+\frac{x^2-9}{x^2-10}.$

18. $2+\frac{2x+1}{2(x-1)}-\frac{4x+5}{2(x+1)}=\frac{x^2+3x+2}{x^2-2x+1}.$

19. $x^2+a(2a-x)-\frac{3b^2}{4}=\left(x-\frac{b}{2}\right)^2+a^2.$

20. $\frac{4-9x}{1-3x}-\frac{5-12x}{7-4x}=2-\frac{24x^2-5}{7-25x+12x^2}.$ 2

21. $\frac{2x+3a}{x+a}=\frac{2(3x+2a)}{3x+a}.$ a

22. $\frac{2(x-b)}{3x-c}=\frac{2x+b}{3(x-c)}.$

(*c*)

Solve the following equations :—

1. $\frac{7}{x-6}+\frac{\overset{3}{8}}{x-11}=\frac{9}{x-7}+\frac{1}{x-12}.$ 13

2. $\frac{5}{x-6}+\frac{4}{x-9}=\frac{8}{x-7}+\frac{1}{x-9}.$ -7

3. $\frac{2x+7}{3x-7}+\frac{3x-6}{2x-5}+\frac{5(x-1)}{9x-25}=\frac{3x-2}{2x-5}+\frac{5x-8}{9x-25}+\frac{2x+2}{\underset{3}{\cancel{5}}x-7}.$ 3

4. $\frac{3x-7}{2x-9}-\frac{3(x+1)}{2(x+5)}=\frac{11x+3}{2x^2-3x-27}.$ $4\frac{1}{2}$

5. $\frac{x+4}{2x+4}+\frac{1}{x+4}+\frac{x}{2x-4}=\frac{x(x+6)}{x^2+2x-4}-\frac{1}{x}.$ -1

6. $\frac{a}{b+cx}-\frac{b}{a+cx}=\frac{a-b}{c+cx}.$

7. $\frac{a+b}{a-b}x-\frac{a+c}{a-c}x+\frac{b-c}{b+c}(x-a)=0.$

8. $\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x-a}+\frac{1}{x-b}=\frac{4x}{x^2-\frac{1}{2}(a^2+b^2)}$. 0

9. $\frac{3}{4}+\frac{12-11x}{8x+20}-\frac{5x+2}{12x+30}=\frac{8-x}{6x+15}-\frac{1}{3}$. 5

10. $\frac{1}{a}\left\{x-\frac{1}{b}(x+c)\right\}=\frac{1}{b}\left\{x-\frac{1}{a}(x+c)\right\}$. $x = 0$ unless $b-$

11. $\frac{1}{x+1}+\frac{1}{x-1}+\frac{1}{x+3}+\frac{1}{x-3}=\frac{4x}{x^2-5}$. 0

12. $\frac{a}{(x-a)(x-c)}-\frac{c}{(a-c)(a-x)}+\frac{x}{(c-x)(c-a)}=\frac{1}{x-c}$.

13. $\frac{x^2+2x+2}{x+1}+\frac{x^2+8x+20}{x+4}=\frac{x^2+4x+6}{x-2}+\frac{x^2+6x+12}{x+3}$. $x = 0,$

14. $\frac{a(x-a)}{b+c}+\frac{b(x-b)}{c+a}+\frac{c(x-c)}{a+b}=x$. $a+b+c$

15. $\frac{x-b^2+2ac}{a+c}-\frac{x-a^2+2bc}{b+c}=\frac{x-c^2-2ab}{a-b}$. $a^2+b^2+c^2$

16. $\frac{1-x}{1-a}-\frac{1-x}{1-a^2}+\frac{1-x}{1-a+a^2-a^3}-4=\frac{\overset{x-1}{\cancel{1-x}}}{1+a}-\frac{1-x}{1+a^2}-\frac{1-x}{1+a+a^2+a^3}$. $x =$

17. $\frac{a}{x-m}+\frac{b}{x-n}+\frac{c}{x-p}=\frac{a}{x-n}+\frac{b}{x-p}+\frac{c}{x-m}$.

18. $\frac{2}{x-1}-\frac{3}{x+2}=\frac{4}{7(x-3)}-\frac{11}{7(x+4)}$.

19. $\frac{x-9}{x-5}-\frac{x-7}{x-2}-\frac{x-9}{x-4}=\frac{x-8}{x-5}-\frac{x-7}{x-4}-\frac{x-8}{x-2}$. $x = 8$

20. $$\frac{a(m-q)}{x-n}+\frac{b(m-q)}{x-p}+\frac{a(n-m)+b(p-m)}{x-q}$$
$$=\frac{a(n-q)+b(p-q)}{x-m}.$$

21. $$\frac{m(a-b)+c(m+n)}{x-a}-\frac{n(a-b)+c(m+n)}{x-b}$$
$$=\frac{m(a-b)}{x-(a+c)}-\frac{n(a-b)}{x-(b-c)}.$$

22. $$\frac{c(a+b)+a^2}{x-a}-\frac{c(a+b)-b^2}{x-b}=\frac{a^2}{x-(a+c)}+\frac{b^2}{x-(b-c)}.$$

23. $$(a-r)\left(\frac{x}{n-o}-\frac{1}{p-q}\right)=(b-r)\left(\frac{x}{n-o}-\frac{1}{p-q}\right).$$

24. $$\frac{x-2a}{b+c-a}+\frac{x-2b}{c+a-b}+\frac{x-2c}{a+b-c}=3.$$

25. $$\frac{a-x}{a^2-bc}+\frac{b-x}{b^2-ac}+\frac{c-x}{c^2-ab}=\frac{3}{a+b+c}.$$

136. We now give some additional problems to be solved by simple equations; the student, it is hoped, will now have but little difficulty in *translating them into the language of algebra*, *i. e.* into equations, and so solving them with facility. A few examples are worked out.

Ex. 1. In going a certain distance a train travelling 35 miles an hour takes two hours less than one travelling 25 miles an hour. Find the distance.

Let x = distance in miles; then $\frac{x}{35}$ = time taken by faster train, and $\frac{x}{25}$ = time taken by slower train, and this is two hours more then the other, *i. e.*

$$\frac{x}{25}-2=\frac{x}{35}\;\therefore\;\frac{x}{25}-\frac{x}{35}=2\;\therefore\;2x=350 \text{ and } x=175.$$

Ex. 2. A number has two digits, of which that in the tens' place is 3 more than the other, and the number itself is 7 times the sum of its digits. Find the number.

Let x denote tens' digit, then $x-3$ = units' digit; and $10x+x-3$ is the No. $\therefore\ 10x+x-3=7(x+x-3)$, or $11x-3=14x-21$.

$$\therefore\ 3x=18, \text{ and } x=6. \quad \therefore \text{ the number is } 63.$$

Ex. 3. 2850 acres are divided between A, B, and C, so that A's share is to B's as 6 : 11, and C has 300 acres more than A and B together; find each man's share.

Let $6x$ denote A's share; then $11x$ will denote B's share, and $6x + 11x + 300 = 17x + 300$ will denote C's share.

$$\therefore 6x + 11x + 17x + 300 = 2850;$$

$\therefore x = 75$, and A's share $= 6 \times 75 = 450$, etc.

Ex. 4. There are two lines of railway between the towns A and B, one 30 miles longer than the other. A train proceeding by the longer route loses one-tenth of the time occupied by the journey. Another train proceeding by shorter route at a rate which is to that of the first as 21 : 20, loses one-fifth of the time occupied by the journey. Both trains accomplish the distance in the same time; find the distance between the towns.

Let $x =$ the No. of miles in shorter distance; then $x + 30 =$ No. in longer distance. Since $\frac{1}{10}$ of actual time was lost in the first case, this is equivalent to *lengthening* the distance *one-ninth; i. e.* $x + 30 + \frac{1}{9}(x + 30)$ miles could have been run in the same time if there had been no stoppages. Second train lost $\frac{1}{5}$ actual time, which may be considered as lengthening the distance *one-fourth*, making it $x + \frac{1}{4}x$; and since the distances are proportional to the *rates*, we have

$$\frac{x + 30 + \frac{1}{9}(x + 30)}{x + \frac{1}{9}x} = \frac{20}{21}, \text{ whence } x = 430.$$

Ex. 5. A question was lost on which 600 persons had voted; the same persons having voted again on the same question, it was carried by twice as many as it was before lost by, and the new majority was to the former as 8 : 7. How many changed their minds?

Let $x =$ No. who voted *against* the question at first.
Then $600 - x =$ No. who voted *for* the question.

And $x - (600 - x) = 2x - 600 =$ No. by which question was lost, then $4x - 1200 =$ No. by which question was carried on second vote.

$\therefore\ \frac{1}{2}\{600-(4x-1200)\}=900-2x=$ total *minority* on 2d vote.

$\therefore\ 600-(900-2x)=2x-300=$ " *majority* " " "

$\therefore\ \frac{2x-300}{x}=\frac{8}{7}$, whence $x=350$ who opposed the question the first time; and $900-2x=200$ who voted against it the second time.

EXERCISE LXIV.

PROBLEMS.

(*a*)

1. A farmer took a certain number of eggs to market for which he asked 36 cents a dozen; by accident 10 of them were broken, but by selling the remainder at the rate of 23 for 75 cents he realized as much as he had expected. How many eggs did he take to market? 125

2. A man has a certain amount of capital; he invests $\frac{2}{5}$ of it so that it brings him 13 per cent, $\frac{1}{3}$ of it in railway stock which pays him 9 per cent, and with the rest he builds a city residence on which he pays for repairs and taxes 3 per cent of its cost; if his net income is $2664, find the amount of his capital. $36000

3. An investment of $4800 is increased to $6972 in a certain number of years; for $\frac{1}{3}$ of the time it brought $3\frac{1}{2}$ per cent, for $\frac{1}{4}$ of the time $3\frac{3}{4}$ per cent, and for the rest of the time 4 per cent. For how long was the capital invested? 2 yrs

4. A student having saved some money attended a High School for a year; he expended one-sixth of his savings in books, $247 for board and other expenses, and at the end of the year he had still one per cent of his savings. What sum had he saved? $300

5. A boy agreed to work a year for a dollars and a suit of clothes, *but after m months he was discharged and received only b dollars and the suit of clothes. Find the value of the suit. $\frac{12b+ma}{-m+12}$

6. A person has only a hours at his disposal; how far may he ride at the rate of b miles an hour, and return on foot at the rate of c miles an hour? $\frac{abc}{b+c}$ mls

7. If the half of a certain number be subtracted from 468, and the remainder be taken from 135, 79 divided by the result will give a quotient $1\frac{35}{44}$. Find the number.

8. There are 2 foremen, 19 mechanics, and 12 labourers engaged at a certain work. Each foreman gets $1.25 more than a mechanic, and each mechanic $1.25 more than a labourer; the total earnings are 111\frac{1}{4}$ a day. Find the daily earnings of a mechanic.

9. A number has six digits, the last digit on the *left* hand being 1; if this digit be brought to the *right* hand, the resulting number is three times the original number. Find the number. 142 857

10. A house cost 3 times as much for materials as for labour; had the materials cost $7\frac{1}{2}$ per cent more, and the labour 5 per cent less, the cost of the house would have been $8350. What was its actual cost? $ 5000

11. A farm is let for $384 and the value of a certain number of bushels of wheat. When wheat is 1.18$\frac{1}{2}$ a bushel the whole rent is 15 per cent lower then when it is $1.75 a bushel. Find the number of bushels of wheat that forms part of the rent.

12. The difference of the squares of two consecutive numbers is 181. Find the numbers.

13. The length of a room exceeds its breadth by 4 feet. If the length were 5 feet more, and the breadth 5 feet less, the area of the room would be 85 feet less. Find the length and breadth.

14. I invested part of $2000, and gained 10 per cent on the investment; but I lost $6\frac{1}{4}$ per cent on the remainder, thereby reducing my gain on the whole $4\frac{4}{5}$ per cent. Find the particular investments.

15. A number consists of 6 digits, the right hand digit being 2; if this be removed to the left hand, the resulting number is only $\frac{1}{3}$ of the original number. What is the number?

16. Three boys have each the same amount of pocket money; the first would spend his in 76 days, the second his in 95 days,

and the third spends 12 cents per day more than the second; their united spendings for 20 days would amount to the entire pocket money each possessed. Find this amount.

17. A man bought a number of eggs at a cent a piece, 4 times as many at 10 cents a dozen, and 5 times as many at 16 cents a dozen, and sells them all at 88 cents a hundred, thereby gaining 84 cents by the transaction. How many eggs did he buy?

18. The fore-wheel of a carriage makes 64 revolutions more than the hind-wheel in travelling a mile; if the circumference of the fore-wheel be increased 11 inches, it will make only 40 revolutions more than the hind-wheel in travelling a mile. Find the circumference of each wheel.

19. The sum of \$17000 is divided among A, B, C, D, and E; B is to have $1\frac{1}{2}$ times as much as A less \$300; C is to have $\frac{3}{4}$ of what A and B get, and \$113 besides; D is to have $\frac{4}{7}$ what A and C get, less $\frac{3}{5}$ of B's share; and E gets $\frac{1}{6}$ of what all the others receive, and \$627 besides. What sum does each receive?

20. In my right pocket I have as many dollars as there are cents in my left pocket; but if I take \$6.93 out of my right pocket, and put it in my left, I shall then have as many dollars in my left pocket as I have cents left in my right. How much money have I in my right pocket?

(*b*)

1. A water-cistern was filled to a certain height; $\frac{5}{12}$ of the contents and 40 gallons were drawn off; then a quantity 20 gallons less than $\frac{4}{13}$ of the remainder was pumped into the cistern, and of what it now contained $\frac{7}{11}$, all but 20 gallons, was drawn off; the cistern now contained 700 gallons less than it contained at first. How many gallons were in it at first?

2. A man had a number of acres of meadow land; he exchanged $\frac{1}{4}$ of it for a certain quantity of vineyard, $\frac{1}{5}$ of it for woodland, and $\frac{1}{6}$ of it for arable land, and he now finds himself possessed of 574 acres in all. If 5 acres of meadow were worth 3 acres of vineyard, and 6 acres of vineyard worth

25 acres of woodland, and 5 acres of woodland worth 4 acres of arable land, how many acres of meadow had the man at first?

3. I have two equal sums of money to pay, one in 9 months, the other in 15 months. I pay both debts immediately, one by \$1208, the other by \$1160. Find the sum of money and the rate of discount (bank discount).

4. If a man gains n per cent when he sells an article at the price p, how much per cent will he gain or lose when his selling price is q?

5. There are three competitors in a mile race; A, who can run the distance in $4\frac{1}{2}$ minutes, gives B a start of 20 yards, and C a start of 60 yards; he reaches the winning post 2 seconds before C, and 4 seconds before B; when and where did A overtake B and C respectively?

6. A and B run a mile race; in the first heat A gives B 20 yards start, and beats him by 30 seconds. In the second heat, he gives B 32 seconds start, and beats him by $9\frac{5}{11}$ yards. Find A's rate per hour?

7. A milkman buys 25 gallons of milk at 20 cents a gallon, but gets short measure; his expenses come to \$1; he adds 5 gallons of water, and sells the mixture at $26\frac{2}{3}$ cents per gallon, gaining $66\frac{2}{3}$ cents. Of how many gallons was the cheating milkman cheated when he bought the milk?

8. A trader allows \$500 for expenses, and augments that part of his capital which is not so expended, by $\frac{1}{3}$ of it; at the end of three years, his stock is doubled. What had he at first?

9. Two trains leave A for B at an interval of $2\frac{1}{2}$ hours, at 19 and 38 miles per hour, respectively; on a certain day the slower train was delayed 50 minutes on the way, and reached B only 10 minutes before the other was due. Find the distance between A and B.

10. A man spends the mth of his income on board and lodging, the nth of it on clothes, the pth of it on amusements, the qth of it in charities, and at the end of the year has \$$r$ left. Find his income.

11. A number has three digits, the hundreds' and the units' digit being each half of the tens' digit; if the tens' figure be interchanged with the hundreds', the resulting number will be greater by 360 than the original number. Find the number. 484

12. Find three consecutive numbers, such that a third of the greatest may be 2 less than the other two together.

13. A person bought a lot of land for \$5400, from which he cut off four-ninths for himself. At a cost of \$200 he made a road which took one-tenth of the remainder, and then sold the rest at the rate of 10 cents a square yard more than double what it cost him, thus clearing \$400 by the transaction. How much land did he buy?

14. Fifteen guineas should weigh 4 ounces; but a certain parcel of light gold having been weighed and counted, was found to contain 9 more guineas than was supposed from the weight, and it appeared that 21 of these coins weighed as much as 20 true guineas. How many were there altogether?

15. A man rides one pth of the distance from A to B at the rate of a miles an hour, and the rest of the distance at the rate of b miles an hour. If he had maintained a uniform rate of b miles an hour, he could have made the distance from A to B and back, in the time occupied by the single journey. Prove that $\frac{1}{a}-\frac{1}{b}=\frac{p}{b}$.

16. A number has four digits, of which the two middle ones are each zero, and the right hand digit is half the left hand digit. Prove that the difference of the squares of the number, and of the number formed by inverting the digits is $749,999\frac{1}{4}$ times the square of the left hand digit.

17. At an army review the troops were drawn up in a solid mass 40 deep, when there were just $\frac{1}{4}$ as many men in front as there were spectators; had the depth been increased by 5 and the spectators drawn up with the army, the number of men in front would have been 100 fewer than before. Find the force of the army.

18. A certain number is divided into n, and also into $n+1$ equal parts; the product of the n equal parts is n times the product of the $n+1$ parts. Find the number.

19. A farmer buys m sheep for \$$p$, and sells n of them at a gain of 5 per cent; how must he sell the remainder that he may gain 10 per cent on the whole?

20. A passenger train which ought to run a certain distance in $2\frac{1}{2}$ hours, is 6 minutes late at the end of 80 miles; by increasing its speed to as many miles an hour as there were miles in half the whole distance, it arrived at its destination on time. Find the speed of the train at first, and the distance run.

21. A student was working with four numbers in proportion; thinking to make the work easier, he subtracted a certain number from each term of the proportion, and so derived the false proportion $41:93::7:51$. Find the true proportion.

22. A man has two kinds of tobacco, which cost, respectively, m cents and n cents a pound; he wishes to take some of each kind, to make up a quantity of a pounds, worth p cents a pound. How many pounds of each sort must he take?

23. What value must be given to x so that the product

$$(a^2+ab+b^2x)(a^2-ab+b^2x)$$

may be equal to $a^4+b^4x^2$?

24. A man wishes to set a number of vines in a rectangular field of which the length is to the breadth as $7:5$; by planting them a certain distance apart, he has 2832 vines remaining; but if he plants them nearer together so that there will be 14 more on the longer side, and 10 more on the shorter side, there will be only 172 remaining. How many vines had he?

CHAPTER XIII.

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

137. In this chapter we shall consider the solution of simple equations involving *more than one unknown* quantity.

Ex. 1. Find two numbers whose sum is 9.

ARITHMETICAL STATEMENT.	ALGEBRAIC STATEMENT.
1st No. + 2d No. = 9.	$x + y = 9.$ where x represents the first number and y, " the second.

It is evident that the number of solutions of the problem is unlimited; we may take *any number whatever* as the "first number," and obtain a corresponding value for the "second number." Thus, we may take

for the first number,	$1, 2, 3, 4, 1\frac{1}{2}, 2\frac{1}{2},$ etc., etc.,
and the second number will be	$8, 7, 6, 5, 7\frac{1}{2}, 6\frac{1}{2},$ etc., etc.

This is expressed algebraically by saying that the equation

$$x + y = 9$$

is satisfied by any one of the following pairs of values :—

$$\begin{array}{ll} x = 1, & y = 8, \\ x = 2, & y = 7, \\ x = 3, & y = 6, \\ x = 4, & y = 5, \\ x = 1\frac{1}{2}, & y = 7\frac{1}{2}, \\ x = 2\frac{1}{2}, & y = 6\frac{1}{2}, \\ \text{etc.,} & \text{etc.,} \end{array}$$

the number of solutions being unlimited.

Ex. 2. Twice A's age increased by three times B's age is 50 years; find their ages.

ARITHMETICAL STATEMENT.	ALGEBRAIC STATEMENT.
Twice A's age + three times B's age = 50.	$2x + 3y = 50$, where x represents A's age, and y represents B's age.

It is clear from the arithmetical statement, that *any positive number* may be taken to represent A's age, that 50 *minus* that number will represent B's age, and that, in every case, the sum of twice A's age and three times B's age is 50; so, various values may be given to x in the equation

$$2x + 3y = 50,$$

each of which will give a corresponding value to y.

Thus, A's age is 1 when B's is 16,
A's " " 2 " B's " $15\frac{1}{3}$,
A's " " 3 " B's " $14\frac{2}{3}$, etc.

This is expressed algebraically by saying that the equation

$$2x + 3y = 50$$

is satisfied by any one of the following pairs of values:—

$$x = 1, \quad y = 16,$$
$$x = 2, \quad y = 15\tfrac{1}{3},$$
$$x = 3, \quad y = 14\tfrac{2}{3}, \text{ etc.}$$

138. From a study of the above problems we learn:—

1°. That *a single equation involving the first power of two unknown quantities expresses some* **one** *relation* between these quantities. The equation in Ex. 1 gives as a relation the sum of two quantities; that in Ex. 2, the sum of twice one quantity and three times another.

2°. That **one** *relation between* **two** *quantities is not sufficient to determine definite values* for these quantities; or, in other words,

3°. That there is an *indefinite number of solutions for an equation which expresses but* **one** *relation between* **two** *unknown quantities.*

EXERCISE LXV.

1. Write equations which express the relations given in the following :—

(1.) The difference between John's age and William's age is 5 years. Let x and y represent their ages.

(2.) Three times one number together with one-half another is 22. Let x and y represent the numbers.

(3.) If A's money were increased by \$3.00 he would have three times as much as B. Let x represent A's money and y B's.

(4.) If A were to give \$5.00 to B, they would have equal sums of money. Let x represent A's money and y B's.

(5.) Seven years ago the age of a father was four times that of his son. Let x and y represent their ages *now*.

(6.) A certain fraction becomes 2 when 7 is added to the numerator. Let x represent the numerator, y the denominator.

2. Express in words the relations given by the following equations :—

(1.) $2x + 5y = 7.$ (2.) $3x - 5y = 6.$ (3.) $7x - 8y = 5.$

(4.) $3x + 7 = 2y.$ (5.) $8 - 3x = y.$ (6.) $\frac{x}{2} + \frac{y}{5} = 10.$

3. Find the value of x which will satisfy the following equations for the assigned values of y :—

(1.) $x + y = 10$, when $y = 1, 2, 3$, or $1\frac{1}{2}$.

(2.) $3x + 4y = 10$, when $y = 6\frac{1}{2}, 2$, or $3\frac{1}{4}$.

(3.) $\frac{1}{3}x + \frac{4}{5}y = 6\frac{1}{2}$, when $y = 5, \frac{1}{10}$, or $\frac{1}{25}$.

(4.) $\frac{x + 17}{4} = x + \frac{68}{x + y}$, when $y = 3$.

(5.) $x - \frac{y - a}{b} = c$, when $y = \frac{b^2d + ab - bc + a}{b^2 + 1}$.

4. Find the value of y which will satisfy the following equations for the assigned values of x :—

(1.) $x - \frac{2y - x}{23 - x} = 20 - \frac{59 - 2x}{2}$, when $x = 21$.

(2.) $\frac{6x^2 - 24y^2 + 130}{2x - 4y + 3} = 3x + 6y + 1$, when $x = 9$.

(3.) $\frac{9xy - 110}{3y - 4} + \frac{151 - 16x}{4y - 1} = 3x$, when $x = 9$.

(4.) $m - \frac{ny}{x} = \frac{d}{x}$, when $x = \frac{nc + bd}{na + mb}$.

139. Ex. 3. The sum of two numbers is 9, and their difference is 1 ; find the numbers.

ARITHMETICAL STATEMENT.	ALGEBRAIC STATEMENT.
1st No. + 2d No. = 9,	$x + y = 9$,
1st No. − 2d No. = 1.	$x - y = 1$.

It is evident from the arithmetical statement that, although there are any number of pairs of numbers whose sum is 9, and any number of pairs whose difference is 1, there are *only two* numbers, 5 and 4, whose sum is 9 *and* difference 1 ; hence, although there are an indefinite number of pairs of values which will satisfy the equations,

$$x + y = 9,$$
$$x - y = 1,$$

when each equation is considered *by itself*, yet there is only one pair of values,

$$x = 5, \ \ y = 4,$$

which will satisfy the equations when the *same* pair of values has to satisfy *both* equations.

Equations that are to be satisfied by the *same* values of the unknown quantities involved are called **Simultaneous Equations.**

140. It has been already noticed that each equation involving the first power of two unknown quantities expresses some *one relation* between these quantities: notice now that the above equations,

$$\begin{aligned} x + y &= 9, \\ x - y &= 1, \end{aligned}$$

express *different* relations between the *same* two quantities, the one expressing the *sum*, the other, the *difference* of the quantities represented by x and y.

Equations that express *different* relations between the unknown quantities involved are called **Independent Equations.**

The equations

$$\begin{aligned} 2x + 3y &= 8, \\ 3x + 2y &= 4, \end{aligned}$$

are *independent equations*, because *neither of the two given relations* can *be inferred from the other.*

But the equations

$$\begin{aligned} 2x + 2y &= 16, \\ x + y &= 8, \end{aligned}$$

are *not independent*, because they express the *same relation* between the quantities involved. The second equation can be derived from the first by dividing both its members by 2.

Hence, we infer that, in order that it may be possible to determine *definite* values for the unknown quantities involved in simultaneous equations, *there must be as many independent equations as there are unknown quantities.*

141. Elimination.—The values of the unknown quantities involved in simultaneous equations are determined by combining the equations in such a manner as *to cause one or more of the unknown quantities to disappear*, and thus to obtain a *single equation* containing but *one unknown* quantity. This process is called **Elimination.**

There are *four* general methods of elimination.

I. By **Addition or Subtraction** (*Cross Multiplication*).
II. By **Substitution.**
III. By **Comparison.**
IV. By **Arbitrary Multipliers.**

We shall illustrate these methods, first in the case of *two unknowns*, and then in the case of *three or more unknowns.*

142. I. Addition or Subtraction.—The following solutions will illustrate the method of *elimination by Addition or Subtraction.*

Ex. 1. Solve $x + y = 10$ (1)

$x - y = 6$ (2)

(1) + (2) gives $2x = 16.$ (3)

Thus an equation has been obtained which does not contain y, *i. e.*, y has been *eliminated* between the equations (1) and (2). The value of x can be found by solving equation (3).

$$2x = 16; \quad \therefore \; x = 8.$$

The value of y can be found by *substituting* this value for x in either of the given equations.

Thus $x + y = 10,$

or $8 + y = 10; \quad \therefore y = 2.$

The value of y could also be determined by *subtraction.*

Thus (1) − (2) gives $2y = 4; \quad \therefore y = 2.$

Ex. 2. Solve $4x + 3y = 42$ (1)

$4x - 7y = 30$ (2)

(1) − (2) gives $10y = 12; \quad \therefore y = 1\frac{1}{5}.$

Substitute this value for y in (2);

$$\therefore 4x - 7 \times 1\tfrac{1}{5} = 30,$$

or $4x = 38\frac{2}{5}; \quad \therefore x = 9\frac{3}{5}.$

Notice that the equations are to be *added* to eliminate a quantity, when the coefficients of that quantity are alike in value but different in sign; and that they are to be *subtracted*, when the coefficients are alike in value and in sign.

EXERCISE LXVI.

1. Define Simultaneous Equations. Write simultaneous equations which express the relations given in each of the following:—

i. The sum of two numbers is 11 and their difference is 1. Let x represent one number and y the other.

ii. The sum of 7 times the greater of two numbers and 3 times the less is 27, and the product of their difference into 5 is 5. Let x represent the greater and y the less.

iii. If to the sum of the ages of two persons 18 be added, the result will be double the age of the elder; but if 6 be taken from the difference of their ages the remainder will be the age of the younger. Let x represent the age of the elder, y the age of the younger.

iv. If the numerator of a fraction be doubled and the denominator increased by 3, its value will be $\frac{2}{5}$; but if the denominator be doubled and the numerator increased by 5, its value will be $\frac{1}{2}$. Let x represent the numerator, y the denominator.

2. What is meant by Independent Equations? Which of the following pairs of simultaneous equations are independent?

i. $5x + 6y = 76$, $\quad 4x - 3y = 14.$
ii. $3x + 2y = 7$, $\quad 6x + 4y = 14.$
iii. $3x - 5y = 4$, $\quad 2x + 3y = 20.$
iv. $4x - 7y = 6$, $\quad 2x - 3 \cdot 5y = 3.$

3. Define Elimination. Eliminate x between each of the following pairs of simultaneous equations:—

i. $x + y = 12$, $\quad y - x = 4.$
ii. $3x - 2y = 5$, $\quad 3x + 3y = 15.$
iii. $3y - 7x = 8$, $\quad 7x + y = 12.$
iv. $6x + 3y = 27$, $\quad 2y + 6x = 26.$

4. Eliminate y between each of the following:—

i. $3x - 4y = 13$, $\quad 2x - 4y = 6.$
ii. $4x - 3y = 6$, $\quad -5x + 3y = -9.$
iii. $7x - 4y = -1$, $\quad 4y - 2x = 6.$

5. Solve:—

i. $x + y = 4$, $\quad x - y = 1.$
ii. $6x + 3y = 27$, $\quad 2y + 6x = 26.$
iii. $17x - 13y = 21$, $\quad 2x + 13y = 17.$
iv. $15y - 2x = 43$, $\quad 4x + 15y = 49.$

143. If the coefficients of the quantity to be eliminated be not equal, they can be made so by either multiplying or dividing one or both equations by suitable numbers; these can usually be determined by inspection. Or, to get the smallest multipliers, we *find the L. C. M. of the coefficients of the quantity to be eliminated; this divided by each of these coefficients will give the required multiplier*

Ex. 3. Solve $$2x + 3y = 8. \qquad (1)$$

$$3x + 7y = 7. \qquad (2)$$

Multiply (1) by 3 and (2) by 2, and we obtain

$$6x + 9y = 24 \qquad (3)$$
$$\underline{6x + 14y = 14} \qquad (4)$$

(3) – (4) gives $$-5y = 10;$$

$$\therefore \quad y = -2.$$

Substitute this value for y in (1);

$$\therefore \quad 2x - 6 = 8;$$

$$\therefore \quad x = 7.$$

Ex. 4. Solve $$12x + 15y = 39, \qquad (1)$$

$$18x - 17y = 19. \qquad (2)$$

L. C. M. of 12 and 18 is 36; $36 \div 12 = 3$, the multiplier for (1); and $36 \div 18 = 2$, the multiplier for (2). Then,

(1) × 3 gives $$36x + 45y = 117 \qquad (3)$$
(2) × 2 gives $$\underline{36x - 34y = 38} \qquad (4)$$
(3) – (4) gives $$79y = 79;$$

$$\therefore \quad y = 1.$$

Substitute this value for y in (2);

$$\therefore \quad 18x - 17 = 19;$$

$$\therefore \quad x = 2$$

144. Sometimes the given equations may be added and subtracted in order to obtain easier forms.

Ex. 5. Solve $2x + 3y = 12,$ (1)

$3x + 2y = 13.$ (2)

(1) + (2) gives $5x + 5y = 25;$

or, $x + y = 5,$ (3)

(1) − (2) gives $-x + y = -1,$ (4)

(3) − (4) gives $2x = 6;$

$\therefore \quad x = 3.$

(3) + (4) gives $2y = 4;$

$\therefore \quad y = 2.$

EXERCISE LXVII.

Solve, by Addition or Subtraction :—

1. $3x - y = 7,$ $3y - x = 3.$ 3, 2
2. $2x + 3y = 0,$ $x - y = 10.$ 6 −4
3. $4x - y = 1,$ $2x + 3y = 11.$ 1, 3
4. $3x + 2y = 23,$ $2x + 3y = 22.$ [illegible]
5. $4x + 5y = 1,$ $5x - 4y = 32.$ 1, −3
6. $9x - 2y = -24,$ $2x - 9y = -31.$ −2, 3
7. $32x - 25y = -18,$ $24x + 26y = 76.$ 1, 2
8. $15x - 16y = -3\frac{1}{6},$ $12x - 13y = 1\frac{1}{3}.$ $-20\frac{5}{6}, -19\frac{1}{3}$
9. $18x - 19y = 9,$ $16x - 17y = 7.$ 10, 9
10. $196x - 208y = 45\frac{1}{3},$ $208x - 196y = 84\frac{1}{3}.$ $\frac{5041}{202}, \frac{1775}{1212}$

145. II. Substitution.—The following solution will illustrate the method of elimination *by substitution.*

Ex. 6. Solve $3x - 4y = 2,$ (1)

$7x - 9y = 7.$ (2)

From (1), $$3x = 2 + 4y;$$

$$\therefore \quad x = \frac{2 + 4y}{3}. \tag{3}$$

Substitute this value for x in (2);

$$\therefore \quad 7\left(\frac{2 + 4y}{3}\right) - 9y = 7;$$

or $$14 + 28y - 27y = 21,$$

$$\therefore \quad y = 7.$$

Substitute this value for y in (3);

$$\therefore \quad x = \frac{2 + 28}{3} = 10.$$

EXERCISE LXVIII.

Solve, by Substitution :—

1. $11x - 9y = -5, \quad x + 2y = 8.$ $x = 2, \ y = 3$
2. $15x + 6y = 16, \quad 5x - y = 10.$ $x = \frac{26}{45}, \ y = -\frac{1}{}$
3. $21x + 8y = 47, \quad 3x + 5y = -1.$ $3, -2$
4. $7x + 6y = 5, \quad 6x + 7y = -5.$ $5, -5$
5. $12x - 7y = 20, \quad 7x + 3y = 40.$ $4, 4$
6. $2x - 13y = 21, \quad 5x + 3y = 13.$ $\frac{232}{71}, -\frac{79}{71}$
7. $9x - 16y = -5, \quad 6x + 5y = 28.$ $3, 2$
8. $21x + 8y = 50, \quad 14x - 5y = 23.$ $2, 1$
9. $20x + 12y = 44, \quad 12x - 20y = 108.$ $4, -3$
10. $9x - 15y = 22\frac{1}{2}, \quad 12x - 7y = 36\frac{1}{2}.$ $3\frac{1}{3}, \frac{1}{2}$

146. III. Comparison.—The following solutions will illustrate the method of elimination *by comparison*.

Ex. 7. Solve $$3x - 2y = 1, \tag{1}$$

$$3y - 4x = 1. \tag{2}$$

From (1), $$3x = 1 + 2y;$$

$$\therefore \quad x = \frac{1 + 2y}{3}. \qquad (3)$$

From (2), $$-4x = 1 - 3y;$$

$$\therefore \quad x = \frac{3y - 1}{4}. \qquad (4)$$

But the values of x obtained in (3) and (4) must be equal, that is,

$$\frac{1 + 2y}{3} = \frac{3y - 1}{4},$$

or $$9y - 3 = 4 + 8y;$$

$$\therefore \quad y = 7.$$

Substitute for y in (3), and

$$x = \frac{1 + 14}{3} = 5.$$

EXERCISE LXIX.

Solve, by Comparison :—

1. $18x - 5y = 55,$ $\quad 6x + 10y = 100.$ $\quad$ 5, 7
2. $12x - 11y = 23,$ $\quad 3x - 15y = 18.$ $\quad$ 1, -1
3. $132x + 119y = -93,$ $\quad 11x - 13y = 61.$ $\quad$ 2, -3
4. $19x + 15y = 31,$ $\quad 21x + 13y = 15.$ $\quad$ $-2\frac{21}{34}$; $5\frac{13}{34}$
5. $11x + 9y = 19,$ $\quad 9x + 7y = 17.$ $\quad$ 5, -6
6. $4x + 3y = 1\frac{7}{18},$ $\quad 3x + 2y = 1.$ $\quad$ $\frac{2}{9}$, $\frac{1}{6}$
7. $5x + 4y = 16\frac{5}{6},$ $\quad 7x - 5y = -\frac{1}{6}.$ $\quad$ [illegible], $\frac{38}{164}$
8. $108x - 91y = -6,$ $\quad 120x - 104y = -24.$ $\quad$ 6
9. $65x + 54y = 33,$ $\quad 39x - 45y = 132.$ $\quad$ $1\frac{3}{5}$, $-1\frac{5}{13}$
10. $2x - 3y = 17,$ $\quad 5x + 7y = -9.$ $\quad$ $3\frac{5}{29}$, $-3\frac{16}{29}$

147. VI. Arbitrary Multipliers.—In this method we multiply one of the given equations *by an arbitrary multiplier* (m, say), add the resulting equation to the other, and then equate to zero the coefficient of y to find x, or the coefficient of x to find y.

Ex. 8.

$$7x - 4y = 20 \tag{1}$$

$$3x + 5y = 22 \tag{2}$$

Multiply (2) by m,

$$\therefore 3mx + 5my = 22m \tag{3}$$

(1) + (3) gives

$$(7 + 3m)x + (5m - 4)y = 20 + 22m. \tag{4}$$

Equation (4) is true for *all values of* m, and therefore for

$$5m - 4 = 0, \text{ or } m = \tfrac{4}{5};$$

in which case the coefficient of y is zero, and equation (4) becomes

$$(7 + \tfrac{12}{5}x) = 20 + 22 \times \tfrac{4}{5}; \text{ and } \therefore x = 4.$$

Similarly, by putting $7 + 3m = 0$, or $m = -\frac{7}{3}$, we eliminate x, and obtain $y = 2$.

Ex. 9.

$$ax + by = c \tag{1}$$

$$mx + ny = p \tag{2}$$

(2) × l gives

$$lmx + lny = lp \tag{3}$$

(1) + (3) gives

$$(a + lm)x + (b + ln)y = c + lp. \tag{4}$$

Assume $b + ln = 0$, or $l = -(b \div n)$, and (4) becomes

$$(an - bm)x = cn - bp; \text{ or } x = (cn - bp) \div (an - bm).$$

Similarly, eliminating x, by putting $a + lm = 0$, we get

$$y = (ap - cm) \div (an - bm).$$

148. Literal simultaneous equations may be solved by any one of the methods of elimination already explained.

Ex. 10. Solve

$$ax + by = m, \tag{1}$$

$$cx + dy = n. \tag{2}$$

(1) × c gives $\quad acx + bcy = cm \quad (3)$

(2) × a gives $\quad acx + ady = an \quad (4)$

(3) − (4) gives $\quad (bc - ad)y = cm - an$

$$\therefore y = \frac{cm - an}{bc - ad}.$$

(1) × d gives $\quad adx + bdy = dm \quad (5)$

(2) × b gives $\quad bcx + bdy = bn \quad (6)$

(5) − (6) gives $\quad (ad - bc)x = dm - bn$

$$\therefore x = \frac{dm - bn}{ad - bc}.$$

N. B.—The value of x might have been more easily determined by symmetry, by interchanging the coefficients of x and y; that is a into b, and b into a, c into d, and d into c.

Thus, $\quad y = \dfrac{cm - an}{bc - ad};$

$$\therefore x = \frac{dm - bn}{ad - bc}.$$

Ex. 11. Solve $\quad a(x + y) + b(x - y) = 1, \quad (1)$

$\quad a(x - y) + b(x + y) = 1 \quad (2)$

(1) + (2) gives $\quad 2ax + 2bx = 2,$

or $\quad (a + b)x = 1 \therefore x = 1 \div (a + b).$

(1) − (2) gives $\quad 2ay - 2by = 0,$

or $\quad (a - b)y = 0; \therefore y = 0.$

EXERCISE LXX.

Solve :—

1. $x + y = a, \quad x - y = b.$ $\quad x = \frac{a+b}{2} \quad y = \frac{a-}{2}$
2. $ax + by = 1, \quad x + y = b.$ $\quad x = \frac{1-b^2}{a-b} \quad y = \frac{ab-}{a-}$
3. $mx - ny = p, \quad nx - my = q.$ $\quad x = \frac{mp-nq}{m^2-n^2} \quad y = \frac{np}{m}$
4. $ax - by = a - b, \quad ax + by = a + b.$ $\quad x = 1 \quad y = 1$
5. $x + ay = b, \quad x + by = a.$ $\quad x = a+b, \quad y = -1$

6. $3ax - 2by = ab, \qquad 2bx - ay = 2b^2 - 3a^2.$ $\frac{b(4b^2-7a^2)}{4b^2-3a^2}, \frac{a(9a^2-}{3a^2-}$

7. $ax + by = 2, \qquad bx + ay = 2.$ $x = y = \frac{2}{a+b}$

8. $(m+n)x - (m-n)y = 4mn,$ $x = m + n$
$(m-n)x + (m+n)y = 2(m+n)(m-n).$ $y = m - n$

x 9. $(a+b)x + (c-b)y = m, \qquad (c+d)x + (a-d)y = m.$ $x = y =$

x 10. $(a-b+c)x + (b-c+a)y = 2a^2,$ $x = \frac{a(bc-2ac-c^2-a}{2(2ab+bc-ac}$
$(a+b+c)x + (c+a-b)y = a^2 + ac.$
$y = \frac{a(3ab+bc+a^2-c^2)}{2(2ab-ac+bc-c^2)}$

149. Equations that are not in their simplest forms should, in general, be simplified before they are combined; but sometimes it is convenient to combine them before simplifying. The student should always be on the watch for any legitimate method of shortening his work.

Ex. 12. Solve

$$\frac{x}{5} + \frac{y}{6} = 18 \qquad (1)$$

$$\frac{x}{2} - \frac{y}{4} = 21 \qquad (2)$$

(6) × 30 gives $\quad 6x + 5y = 540 \qquad (3)$

(2) × 12 " $\quad 6x - 3y = 252 \qquad (4)$

(3) − (4) " $\quad 8y = 288$, and $\therefore y = 36;\ x = 60.$

Ex. 13. Solve

$$\tfrac{1}{8}(x+y) + \tfrac{1}{6}(x-y) = 5 \qquad (1)$$

$$\tfrac{1}{4}(x+y) - \tfrac{1}{3}(x-y) = 10 \qquad (2)$$

(1) × 2 gives $\quad \tfrac{1}{4}(x+y) + \tfrac{1}{3}(x-y) = 10 \qquad (3)$

(2) + (3) " $\quad \tfrac{1}{2}(x+y) = 20$

$$x + y = 40 \qquad (4)$$

(3) − (2) gives $\quad \tfrac{2}{3}(x-y) = 0$

or, $\quad x - y = 0$; and $\therefore x = y = 20.$

Ex. 14. Solve

$$\frac{6x+9}{4} + \frac{3x+5y}{4x-6} = 3\frac{1}{4} + \frac{3x+4}{2}, \qquad (1)$$

$$\frac{8y+7}{10} + \frac{6x-3y}{2y-8} = 4 + \frac{4y-9}{5}. \qquad (2)$$

Taking together all the fractions in (1) having only numerical denominators we get,

$$\frac{3x+5y}{4x-6} = \frac{13+6x+8-6x-9}{4},$$

$$\therefore \frac{3x+5y}{4x-6} = 3,$$

$$\therefore 12x - 18 = 3x + 5y, \text{ or } 9x - 5y = 18. \qquad (3)$$

Taking together all the fractions in (2) having only numerical denominators, we get,

$$\frac{6x-3y}{2y-8} = \frac{40+8y-18-8y-7}{10},$$

$$\therefore \frac{6x-3y}{2y-8} = \frac{3}{2}.$$

$$\therefore 12x - 6y = 6y - 24, \text{ or } x - y = -2. \qquad (4)$$

From (3) and (4) we get

$$y = 9; \quad x = 7.$$

EXERCISE LXXI.

(a)

Solve :—

x 1. $\frac{x}{2} + \frac{y}{3} = 7, \qquad \frac{x}{3} + \frac{y}{2} = 8.$ x = 6, y = 12

2. $\frac{1}{3}(21 - 6y) = \frac{1}{12}(8 + x), \qquad \frac{1}{2}(23 - 5x) = \frac{1}{6}(y + 6).$ 4, 3

3. $\frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4}, \qquad \frac{2y+4}{3} - \frac{2x+y}{8} = \frac{x+13}{4}.$ 7, 10

4. $\frac{2}{3}x + \frac{3}{5}y = 5\frac{4}{5}, \qquad \frac{3}{5}x + \frac{2}{3}y = 4\frac{3}{5}.$ $13\frac{2}{19}$, $-4\frac{17}{19}$

5. $\frac{11x - 5y + 16}{22} = \frac{3x + y + 2}{32}, \qquad 5y - 8x = 12.$ 6, 12

x-6. $\frac{y}{3} - \frac{y}{2} = 2, \qquad \frac{3-2x}{5} - \frac{4+5y}{11} = 4.$ $4\frac{5}{22}$, -12

7. $7\left(\frac{x}{2} - 1\right) - 3(2y + 8) = 0$, $\frac{2x + 2}{9} - \frac{10y + 9}{2} = 0$. $8, -\frac{1}{2}$

8. $\frac{2}{3}x - 4 + \frac{y}{2} + x = 8 - \frac{3y}{4} + \frac{1}{12}$, $\frac{y}{6} - \frac{x}{2} + 2 = \frac{1}{6} - 2x + 6$. $2,$

9. $2x - \frac{\frac{y}{3} + 1}{\frac{4}{3}} = 7 + \frac{1\frac{1}{2}y - x}{2\frac{1}{2}}$, $4y - \frac{2 - \frac{x}{4}}{\frac{3}{4}} = 24\frac{1}{2} - y - \frac{1}{2}$. $x = y$

10. $5x + 4y = 38\frac{1}{4} + \frac{1}{4}(3x - y)$, $x = 5\frac{5}{24} - \frac{1}{4}\left\{\frac{1}{2}(x + y) - \frac{1}{3}(x - y)\right\}$. $4,$

11. $\frac{1}{2}(x + y) + \frac{1}{3}(x - y) = 7$, $\frac{1}{4}(x + y) - \frac{1}{9}(x - y) = 1$. $8\frac{1}{2}, -\frac{1}{2}$

sym 12. $\frac{a}{b}x + \frac{b}{a}y = ab$, $\frac{b}{a}x + \frac{a}{b}y = ab$. $x = y = \frac{a^2b^2}{a^2 + b^2}$

× 13. $\frac{m - n}{m + n}x + \frac{n + m}{n - m}y = 1$, $\frac{n + m}{n - m}x + \frac{n - m}{n + m}y = 1$. $y = \frac{n^4 - n}{n^4 + 6m^2n^2}$

$x = 2mn(n^2 - m^2) \div n^4 + 6m^2n^2 + m^4$

14. $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x - a}{b} + \frac{y - b}{a} = 1$. $\frac{a^2}{a - b}, \frac{b^2}{b - a}$

15. $\frac{x - y}{x + y} = \frac{3}{5}$, $\frac{x}{1 + y} = 2$. $4, 1$

16. $\frac{2x}{x + y} = 3$, $\frac{2y + 8}{x + y} = 1$. $6, -2$

17. $\frac{2x - 3y}{3x - 4y} = \frac{8}{11}$, $\frac{3x - 2y}{2 + 2x - 4y} = \frac{7}{9}$. $-2, 4$

18. $\frac{1 + x}{1 - y} = \frac{4}{5}$, $\frac{2 + x}{2 - y} = \frac{5}{6}$. $3, -4$

(b)

1. $\frac{1 + x + y}{1 - x - y} = \frac{9}{7}$, $\frac{1 + x - y}{1 - x + y} = 1$. $x = y = \frac{1}{16}$

2. $\frac{1 + \frac{x}{7}}{1 + \frac{y}{7}} = -3$, $\frac{\frac{7}{x} - 1}{\frac{7}{x} + \frac{y}{x}} = 1$. $14, -14$

3. $\dfrac{x-3}{y-4}=\dfrac{2x-5}{2y-1}$, $\dfrac{2x-1}{3y+4}=\dfrac{4x-5}{6y-8}$. $4\frac{1}{31}$, $11\frac{7}{31}$

4. $x-\dfrac{2y-x}{23-x}=20-\dfrac{59-2x}{2}$, $y+\dfrac{y-3}{x-18}=30-\dfrac{73-3y}{3}$.

$x=21$ $y=20$

5. $1+\frac{5}{6}(y+5)-\frac{1}{3}(7x-6)=10-\frac{1}{12}(3x-10+7y)$,

$$\frac{\frac{1}{9}(12-x)}{5x-\frac{1}{3}(14+y)}=\frac{1}{8}.$$ 3, 7

6. $\dfrac{x-y+1}{x-y-1}=a$, $\dfrac{x+y+1}{x+y-1}=b$. $\dfrac{ab-1}{(1-a)(1-b)}$; $\dfrac{a-b}{(1-a)(1-b)}$

7. $1-\dfrac{ax-b}{by-c}=0$, $a-\dfrac{2y-3}{1-x}=1$. $y=\dfrac{a^2-2[illegible]+a[illegible]-ab+b-[illegible]}{a[illegible]-b+2a}$

$x=(ab+4b-2c)\div ab-b+2a$

8. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{5}{2}$ $x^2+y^2=90$. ± 9, ± 3

9. $\dfrac{x-2a}{x-3a}+\dfrac{y-4b}{y-3b}=2$, $\dfrac{x+2a}{x+a}=\dfrac{y+5b}{y+3b}$. a, b

10. $\dfrac{1\frac{1}{2}x^2-6y^2+32\frac{1}{2}}{\frac{1}{2}x-y+\frac{3}{4}}=3x+6y+1$, $\dfrac{3xy-36\frac{2}{3}}{y-1\frac{1}{3}}+\dfrac{37\frac{3}{4}-4x}{y-\frac{1}{4}}=3x$. 9,

11. $\dfrac{x}{a+b}-\dfrac{y}{a-b}=\dfrac{1}{a+b}$, $\dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{1}{a-b}$. $\dfrac{a}{a-b}$, $\dfrac{b}{a+}$

12. $\dfrac{x-y}{x-2y}-\dfrac{x-2y}{x-3y}=\dfrac{x-4y}{x-5y}-\dfrac{x-5y}{x-6y}$, $2x+3y=22$. 8, 2

13. $a(2x-y)+b(2x+y)=c(2x-y)+d(2x+y)=1$.

$x=(b+c-a-d)\div 4(bc-ad)$ $y=([illegible])\div 2(bc-ad)$

14. $x+1:x+5:x-1::y+3:y+11:y-1$. Indeterminate

$-2\frac{1}{5}$, $-6\frac{1}{5}$

15. $x-2:x-1:x+4::y-2:y+1:y+4$. $x=y=3\frac{1}{2}$

16. $x-8:x-4:x-12::y-12:y:y-24$. $6\frac{2}{3}$, 8 Indeterm

17. Solve $ax+by=c$, $a'x+b'y=c'$; determine and explain the values of x when

(1.) $\dfrac{a}{a'}=\dfrac{b}{b'}$. (2.) $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}$.

$x=\dfrac{b'c-[illegible]}{[illegible]}$ $y=\dfrac{ac'-[illegible]}{[illegible]}$

150. Simultaneous equations containing fractions, the denominators of which are simple expressions containing the unknown quantities, are generally more easily solved by combining the equations before clearing of fractions.

Ex. 15. Solve $$\frac{3}{x} - \frac{4}{y} = 5, \qquad (1)$$

$$\frac{4}{x} - \frac{5}{y} = 6. \qquad (2)$$

(1) × 4, gives $$\frac{12}{x} - \frac{16}{y} = 20 \qquad (3)$$

(2) × 3, gives $$\frac{12}{x} - \frac{15}{y} = 18 \qquad (4)$$

(3) − (4), gives $$-\frac{1}{y} = 2$$

or $$y = -\tfrac{1}{2}.$$

Substitute this value for y in (1),

$$\therefore \quad \frac{3}{x} - \frac{4}{-\frac{1}{2}} = 5;$$

$$\therefore \quad x = -1.$$

Equations which are not given in this form may sometimes be changed into it and solved.

Ex. 16. Solve $$5x + 7y = 19xy, \qquad (1)$$

$$3x - 8y = -7xy. \qquad (2)$$

Divide each equation by xy,

$$\therefore \quad \frac{5}{y} + \frac{7}{x} = 19, \qquad (3)$$

and $$\frac{3}{y} - \frac{8}{x} = -7. \qquad (4)$$

The values of x and y can be found from (3) and (4) in the same way as in the last example.

EXERCISE LXXII.

Solve :—

1. $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$, $\frac{1}{x}-\frac{1}{y}=\frac{1}{6}$. 3, 6

2. $\frac{1}{x}+\frac{2}{y}=4$, $\frac{11}{x}+\frac{3}{y}=25$. $\frac{1}{2}$, 1

3. $\frac{3}{x}-\frac{4}{y}=2$, $\frac{5}{x}-\frac{6}{y}=4$. $\frac{1}{2}$, 1

4. $x+y=xy$, $x-y=2xy$. -2; $\frac{2}{3}$

5. $3x+2y=4xy$, $5x+4y=5xy$. $-\frac{2}{5}$; $\frac{1}{3}$

6. $\frac{15x+12y}{9xy}=-1$, $\frac{4x-5y}{14xy}=1$. $-\frac{1}{2}$, 1

× 7. $\frac{a}{x}+\frac{b}{y}=1$, $\frac{b}{x}+\frac{a}{y}=1$. $x=y=a+b$

8. $bcx+aby=xy$, $acx+bcy=xy$. $\frac{b(a^2-bc)}{a-b}$ a^2

9. $\frac{1}{3x}+\frac{1}{4y}=7$, $\frac{1}{2x}+\frac{1}{3y}=5$. $-\frac{1}{78}$, $\frac{1}{132}$

10. $\frac{a}{bx}+\frac{b}{ay}=1$, $\frac{b}{ax}+\frac{a}{by}=1$. $x=y=\frac{a^2+b^2}{ab}$

151. Three or more Unknowns.—We proceed in a similar way to solve equations in three unknowns. Eliminate *any one* of the quantities between any two of the equations, and the *same* quantity between any other two; then eliminate one of the remaining quantities between the two resulting equations.

Ex. 1. Solve

$$2x+4y+4z=18, \qquad (1)$$

$$3x+3y+2z=17, \qquad (2)$$

$$\underline{5x+6y+5z=32.} \qquad (3)$$

(2) × 2, gives $$6x+6y+4z=34, \qquad (4)$$

(4) – (1), gives $$4x+2y=16. \qquad (5)$$

Again,

(2) × 5 gives $15x + 15y + 10z = 85,$ (6)

(3) × 2 " $10x + 12y + 10z = 64,$ (7)

(6) − (7) " $5x + 3y = 21.$ (8)

From (5) and (8), x is found to be 3, and y, 2.

Substitute these values for x and y in (1);

$$\therefore 6 + 8 + 4z = 18, \text{ or } z = 1.$$

Ex. 2. Solve $\frac{1}{x} + \frac{1}{y} = 2,$ (1)

$\frac{1}{x} + \frac{1}{z} = 3,$ (2)

$\frac{1}{y} + \frac{1}{z} = 3.$ (3)

(1) − (2) gives $\frac{1}{y} - \frac{1}{z} = -1,$ (4)

(3) + (4) " $\frac{2}{y} = 2$, and $\therefore y = 1.$

Substitute this value for y in (1);

$$\therefore \frac{1}{x} + 1 = 2, \text{ or } x = 1.$$

Substitute its value for y in (3);

$$\therefore 1 + \frac{1}{z} = 3, \text{ or } z = \frac{1}{2}.$$

N. B.—The student should be careful to combine such equations as shall require least multiplying, or dividing, in preparing them for elimination.

EXERCISE LXXIII.

(*a*)

Solve :—

1. $x - y + z = 4, \quad y - z + x = 4, \quad z - x + y = 4.$ $x = y = z = 4$
2. $2x + y - z = 11, \quad 2y + z - x = 4, \quad 2z + x - y = -4.$ $2\frac{11}{14}, 4\frac{1}{14}, -1$
3. $2x + 3y - 4z = 4, \quad 3x - 2y + 5z = 0, \quad 3x + y - z = -6.$ $2\frac{1}{2}, -4,$
4. $4x - 5y = 23, \quad 3x + z = 2, \quad 2y - 3z = 6.$ $2, -3, -4$

123 -79 5. $x - 2y + 3z = 16$, $2x + 3y - 4z = -32$, $3x + 4y - 6z = 13$.

3, 7 ×6. $10x + 12y - 13z = -15$, $9x + 6y - 7z = 5$, $6x - 5y + 8z = 65$.

, 5, 6 7. $x + 2y = 14$, $y + 2z = 17$, $2x + z = 14$.

, -2, 5 8. $x + y + z = 2$, $x + 2y + 3z = 10$, $x + 3y + 4z = 13$.

, 2, 3 9. $x + 3y + 6z = 25$, $3x + 6y + z = 18$, $6x + y + 3z = 17$.

$-\frac{2}{41}, -1\frac{3}{41}$ ×10. $x - 2y + z = 6$, $3x + 4y + 5z = 14$, $2x - 3y - 6z = 26$.

21, -3 11. $13x + y = 60$, $12x - y + z = 12$, $11x - 4y - 5z = -36$.

95, -5 2.8 12. $x + y + z = 12$, $10x + y + \cdot 12 = 92 \cdot 1$, $3x = y + z$.

, 16 13. $x + y + z = 24$, $3x = y + z$, $2y = x + z$.

2, -3 14. $x + y = 4$, $y + z = -5$, $z + x = 3$.

, 8 15. $x + y = \frac{2}{3}y + 5$, $z + x = \frac{1}{3}x + 10$, $y + z = \frac{1}{4}z + 12$.

0, -8 16. $x + y = \frac{1}{2}z + 8$, $y + z = \frac{3}{5}x + 4$, $z + x = \frac{1}{3}y - 16$.

$, -4\frac{2}{3}$ 17. $\frac{x+3}{y+3} = \frac{2}{3}$, $\frac{y+2}{y+4} = 4$, $\frac{z+3}{x+4} = \frac{1}{12}$.

$-2\frac{1}{10}$

(*b*)

$\frac{1}{6}, -\frac{1}{3}$ 1. $xy + yz + zx = 5xyz$, $xy + 2yz + 3zx = 19xyz$,
$xy + 4yz + 5zx = 35xyz$.

$\frac{5}{4}$ $\frac{5}{13}$ 2. $x + y = xy$, $x + 2z = 3xz$, $y + 3z = 5yz$.

✓3. $\frac{4xy}{4y + x} = 3$, $\frac{3xz}{2x + z} = 2\frac{2}{3}$, $\frac{2yz}{2y + z} = -3$. $\frac{8}{5}, -\frac{6}{7}, 4$

$\frac{1}{b}, \frac{1}{c}$ 4. $ax + by + cz = 3$, $ax - by + cz = 1$, $ax + by - cz = 1$.

5. $ax + by = ab$, $by + cz = bc$, $ax + cz = ac$.

×6. $x + y + z = 0$, $ax + by + cz = 0$, $bcx + cay + abz = 3abc$.

$a - b$ 7.×$(a + b)x + (b + c)y + (c + a)z = 0$, $x + y + z = 0$,
$(b - c)(c - a)x + (c - a)(a - b)y + (a - b)(b - c)z$
$= 3(a - b)(b - c)(c - a)$.

×8. $\frac{a}{x} + \frac{b}{y} = c$, $\frac{b}{y} + \frac{c}{z} = a$, $\frac{a}{x} + \frac{c}{z} = b$.

×9. $x + y + z = (b + c)x + (c + a)y + (a + b)z = 0$,
$bcx + acy + abz = 1$.

10. $\frac{2}{x}-\frac{3}{y}+\frac{4}{z}=2{\cdot}9,\quad \frac{5}{x}-\frac{6}{y}-\frac{7}{z}=-10{\cdot}4,\quad -\frac{8}{x}+\frac{9}{y}+\frac{10}{z}=14{\cdot}9.$ $z=\frac{5\cdot8}{61}$

11. $\frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=7\frac{3}{5},\quad \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=10\frac{1}{6},\quad \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=16\frac{1}{10}.$ $\frac{1}{2}, \frac{1}{3},$

12. $\frac{x}{a+b}+\frac{y}{b-c}+\frac{z}{c+a}=2c,\quad \frac{x}{a-b}-\frac{y}{b-c}+\frac{z}{c-a}=2a,$

$\frac{x}{a-b}-\frac{y}{b-c}-\frac{z}{c+a}=2a-2c.$ $a^2-b^2, b^2-c^2, c^2-a^2$

13. $\frac{x+y-z}{b+c}=\frac{y+z-x}{c+a}=\frac{z+x-y}{a+b}=a+b+c.$ $z=\frac{1}{2}(a+b+c)(2a+b+$

14. $xy=m(ay+bx),\quad yz=n(cz+dy),\quad xz=p(cz+fx).$ $z=$ [illegible]

$x=$ [illegible]

15. $bx+ay+cz=c^2+2ab,\quad cx+by+az=a^2+2bc,$

$ax+cy+bz=b^2+2ca.$

$z=a^3+$ [illegible]

152. When the number of unknowns and of equations *exceeds three*, the same methods of elimination may be followed. Eliminate any one of the unknowns from *all* the equations, then another of the unknowns from all the resulting equations, and so on.

Ex. 1.

$$2x-3y+2z=13, \tag{1}$$
$$4y+2z=14, \tag{2}$$
$$4u-2x=30, \tag{3}$$
$$5y+3u=32. \tag{4}$$

(1) − (2) gives $2x-7y=-1. \quad (5)$

4 (4) − 3 (3) " $6x+20y=38. \quad (6)$

From (5) and (6), $x=3$, $y=1$; then from (2) and (3), $z=9$, $u=5$.

Ex. 2.

$$7x-2z+3u=17, \tag{1}$$
$$4y-2z+t=11, \tag{2}$$
$$5y-3x-2u=8, \tag{3}$$
$$4y-3u+2t=9, \tag{4}$$
$$3z+3u=33. \tag{5}$$

2 (2) − (4) gives $4y - 4z + 3u = 13.$ (6)

3 (1) + 7 (3) " $35y - 6z - 5u = 107.$ (7)

From (5), (6), (7), y, z, u may be found: $x = 2$, $y = 4$, $z = 3$, $u = 3$, $t = 1$.

Ex. 3.

$$x + y + z = a, \quad (1)$$
$$y + z + u = b, \quad (2)$$
$$z + u + x = c, \quad (3)$$
$$u + x + y = d. \quad (4)$$

Add all the equations, and we get

$$3(x + y + z + u) = a + b + c + d. \quad (5)$$

(5) − 3 (2) gives $3x = a - 2b + c + d$;

and y, z, u may be found by symmetry.

EXERCISE LXXIV.

(a)

1. $7x - 3y = 1$, 4 = x
$11z - 7u = 1$. 9 = y
$4z - 7y = 1$, 16 = z
$19x - 3u = 1$. 25 = u

2. $x + y = 3$,
$y + z = 5$,
$z + u = 7$,
$x + u = 5$.

3. $5x + 4y - 8z = 29$, x = 5
$7x - 3y = 26$, y = 3
$2z + 9u = 38$, z = 1
$11x - 6u = 31$. u = 4

4. $3u - 2y = 2$, x = $-5\frac{2}{11}$
$5x - 7z = 11$, y = $16\frac{5}{11}$
$2x + 3y = 39$, z = $-5\frac{3}{11}$
$4y + 3z = 50$. u = $11\frac{7}{11}$

5. $7u - 13z = 87$, 3 = x
$10y - 3x = 11$, y = 2
$3u + 14x = 57$, z = −4
$2x - 11z = 50$. u = 5

6. $x + 2y + 3z = 4$, x = [illegible]
$3x - 2y - v = 6$, y = −1
$2x - 3z - 3v = 6$, z = 2
$y - 4v = 15$. v = −4

7. $x - 2y = 3$, x = −1
$y - 2z = 4$, y = −2
$z - 2u = 5$, z = −3
$u - 2t = 6$, u = −4
$t - 2x = -3$. t = −5

8. $x - 2y + 3z - 4u = -1$.
$-5x + 6y - 7z + 8u = 18$.
$9x - 10y - 11z + 12u = 4$.
$-13x + 14y + 15z - 16u = -4$.
x = 1, y = 2, z = 3, u = 4

$z = a - b + c, \quad t = b + c - a$ **(b)**

1. $$\begin{aligned} x + y + z &= 3a + b + c, \\ x + y + t &= a + 3b + c, \\ x - z - t &= a + b - c, \\ y + z - t &= 3a - b - c. \end{aligned}$$

$x = a + b + c \quad y = a + b - c$

2. $$\begin{aligned} 3z + 2u - 5y &= 18, \\ 3x + y - 4u &= 9, \\ x + 7z - 6y &= 33, \\ 5z - 2x - 8y + 2u &= 15. \end{aligned}$$

✓3. $$\begin{aligned} x + y + z &= a, \\ y + z + u &= b, \\ z + u + x &= c, \\ u + x + y &= d. \end{aligned}$$

4. $$\begin{aligned} x + y = a, \quad y + z &= b, \\ z + u = c, \quad u + v &= d, \\ v + x &= e. \end{aligned}$$

5. $$\begin{aligned} x + y + z &= a, \\ y + z + u &= b, \\ z + u + v &= c, \\ u + v + x &= d, \\ v + x + y &= e. \end{aligned}$$

6. $$\begin{aligned} x + y + z - u - v &= a, \\ y + z + u - v - x &= b, \\ z + u + v - x - y &= c, \\ u + v + x - y - z &= d, \\ v + x + y - z - u &= e. \end{aligned}$$

7. $$\begin{aligned} 1\tfrac{2}{3}x + 2\tfrac{3}{4}y &= 105, \\ 3\tfrac{4}{5}x + 4\tfrac{5}{6}y &= 317, \\ 5\tfrac{6}{7}z + 6\tfrac{7}{8}u &= 741, \\ 7\tfrac{8}{9}u + 8\tfrac{9}{10}x &= 835. \end{aligned}$$

8. $$\begin{aligned} 2x - y - z + 2u - v &= 3a, \\ 2y - z - u + 2v - x &= 3b, \\ 2z - u - v + 2x - y &= 3c, \\ 2u - v - x + 2y - z &= 3d, \\ 2v - x - y + 2z - u &= 3e. \end{aligned}$$

9. $$\begin{aligned} -x + y + z + t + u &= a, \\ -x + y + z + t + v &= b, \\ -x + y + z + u + v &= c, \\ -x + y + t + u + v &= d, \\ -x + z + t + u + v &= e, \\ -y + z + t + u + v &= f. \end{aligned}$$

10. $$\begin{aligned} &xy + yz + zx = 0, \\ &ayz + bxz + cxy = 0, \\ &bcyz + acxz + abxy \\ &\quad + (a-b)(b-c)(c-a)\,xyz = 0. \end{aligned}$$

153. Problems giving Simultaneous Simple Equations.—We add a few examples of *problems which* can be solved by such equations.

Ex. 1. A store-room holds just 13 bales and 33 casks; 9 casks and 5 bales fill exactly one-third of it. How many casks, and how many bales, separately, will fill it?

Let $x =$ No. of bales required, and $y =$ No. of casks.

Then, 1 bale fills $\frac{1}{x}$th of it, and 1 cask fills $\frac{1}{y}$th of it; hence,

$$\frac{13}{x}+\frac{33}{y}=1 \qquad (1); \qquad \frac{5}{x}+\frac{9}{y}=\frac{1}{3} \qquad (2).$$

(1) × 5 − (2) × 13 gives $\frac{48}{y}=\frac{2}{3}$; $\therefore\ y=72$;

substitute this value for y in (2), $\therefore\ x=24$.

Ex. 2. A number consists of 3 digits, the first and last alike; by interchanging the tens' and units' digits, the number is increased by 54; by interchanging the tens' and hundreds' digits, 9 must be added to 4 times the result to make it equal to the original number. What is the number?

Let x, y, and x be the hundreds', tens', and units' digits, respectively; then,

$100x+10y+x=101x+10y=$ required number,

$100x+10x+y=110x+\ \ y=$ No. from interch'g tens and units.

$100y+10x+x=100y+11x=$ " " " hundreds and tens.

Hence we have $\quad 110x+y=101x+10y+54, \qquad (1)$

and $\qquad 4(100y+11x)+9=101x+10y. \qquad (2)$

By reducing, (1) gives $\quad x-y=6, \quad$ or $\quad x=6+y, \qquad (3)$

and (2) gives $\qquad 130y-19x=-3. \qquad (4)$

In (4) substitute the value of x from (3); $y=1$, $x=7$, and the number is 717.

Ex. 3. There are three numbers such that the product of the first and second equals 3 times their sum: the product of the first and third = 8 times their sum, and 7 times the product of the second and third equals 9 times their sum. Find the numbers.

Let x, y, z, represent the numbers; then,

$$xy=3(x+y), \text{ or, dividing by } 3xy, \ \frac{1}{y}+\frac{1}{x}=\frac{1}{3}, \qquad (1)$$

$$xz = 8(x+z), \text{ or, dividing by } 8xz, \quad \frac{1}{z}+\frac{1}{x}=\frac{1}{8}, \qquad (2)$$

$$7yz = 9(y+z), \text{ “ \quad “ \quad “ } 9yz, \quad \frac{1}{z}+\frac{1}{y}=\frac{7}{9}. \qquad (3)$$

$$(1)-(2)+(3) \text{ gives } \qquad \frac{2}{y}=\frac{71}{72},$$

$$\therefore \quad y=\tfrac{144}{71}; \quad z=\tfrac{144}{41}; \quad x=\tfrac{144}{23}.$$

EXERCISE LXXV.

(*a*)

1. Find the numbers whose sum is 857,142, and difference, 571,428. 714285 [illegible] 857
2. Find two numbers such that the first and half the second is $72\frac{1}{2}$, and the second and half the first is 85. 40, 65
3. Said Willie to Charlie: Give me 5 of your marbles, and I shall have three times as many as you; said Charlie: Give me 2 of yours, and I shall have five times as many as you. How many had each? W 41 ; C 8
4. If I divide the less of two numbers by the greater, the quotient is .21 and the remainder .04162. If I divide the greater by the less, the quotient is 4 and the remainder .742. Find the numbers. 1.234 , 5.678
5. The third of one number and the seventh of another make 58; the third of the latter and the seventh of the former make 42. Find the numbers. 147, 63
6. A number of two digits exceeds, by 4, 12 times the units' digit; if 9 be taken from the number, the result will be expressed by the digits taken in inverted order. 76
7. If both terms of a ratio be increased by 5, the resulting ratio is 9 : 11; if both terms be diminished by 5, the resulting ratio will be 2 : 3. Find the terms of the ratio. 13:17
8. Of a two-figure number the sum of the digits is 10; by inverting the digits a number is formed which is 36 less than the original number. Find it. 73

9. A man has two full casks and an empty one; to fill this he may take the whole of the first and a fifth of the second, or the whole of the second and a third of the first; the three casks hold in all 1440 gallons. How much does each hold?
480, 400, 560

10. Two investments, one at 5 per cent, the other at $4\frac{1}{2}$ per cent, yield \$853.20 in one year; if the first were at $4\frac{1}{2}$ per cent and the second at 5 per cent, the interest for the same time would be \$13.50 less. Find the amount of each investment?
$10260, $7560

11. If the numerator and denominator of a fraction be each increased by 2, the new fraction is $\frac{3}{4}$; if each be diminished by 2, the new fraction is $\frac{1}{2}$. Find the fraction.
$\frac{4}{6}$

12. If a two-digit number is 6 times the sum of its digits, show that the number formed by interchanging the digits, is 5 times their sum.

13. There is a number of two digits; a second number is formed by interchanging the digits; the sum of the two numbers is 187 and their difference is 9. Find the original number.
98 or 89

14. A and B can together lift 550 lbs., B and C, 600 lbs., and C and A, 650 lbs.; what weight can each lift? A 300, B 250, C 35

15. A boy is sent to market with apples, which he is to sell at 8 cents a dozen; having given some of them to a companion he was obliged to sell $\frac{3}{5}$ of the remainder at 9 cents a dozen, and the rest at 7 cents a dozen, in order to get the required amount of money; had he sold $\frac{3}{4}$ of the remainder at 9 cents a dozen, and the rest at 7 cents, he would have realized 2 cents more than the required sum. How many apples had he at first, and how many did he give away?
82 gave away 2

16. A man invests \$12000, part at 5 per cent, part at 4 per cent, and the rest at 3 per cent, receiving annually \$490 interest; the sum invested at 5% is half as much as the other two parts. Find the three investments.
$5000, $3000 [illegible]

17. The number 232 is divided into three parts such that the first with half the sum of the other two, the second with one-third the sum of the other two, and the third with one-fourth the sum of the other two make equal numbers. Find the three parts.
40, 88, 104

18. A number is formed of three digits whose sum is 18; if the digits be reversed, the resulting number is 198 greater than the number; and if the number be divided by 12, the quotient is $6\frac{2}{3}$ times the remainder. Find the number. 486

19. A banker has two kinds of money; it takes a pieces of the first to make a dollar, and b pieces of the second to make a dollar; he is offered a dollar for c pieces. How many of each kind must he give? $\frac{a(b-c)}{b-a}$; $\frac{b(c-a)}{b-a}$

20. Find two numbers whose difference is a times and product b times their sum. $\frac{2b}{1-a}$ $\frac{2b}{1+a}$

21. The reciprocal of the difference of two numbers added to the reciprocal of their sum, is 3; the same, diminished by the reciprocal of their sum, is 1. Find the numbers. $\frac{3}{4}$ & $\frac{1}{4}$

22. A cistern can be emptied by two pipes A and B in 35 minutes, by B and C in 42 minutes, and by C and A in 70 minutes; in what time can (1) each separately, (2) all together, empty the cistern? A 105; B $52\frac{1}{2}$; C 210; A, B, & C 30

(*b*)

1. A man had a certain quantity of wine in each of two casks; he poured as much from the first into the second as this contained; then he poured from the second into the first as much as remained in the first; he next poured from the first into the second as much as was left in the second, and each cask was now found to contain 160 gallons. How many gallons were in each at first? 1st 220 gal 2nd 100

2. In a four-figure number the sum of the first and third digits is equal to the sum of the second and fourth, the difference between the third and fourth is three times the difference between the first and fourth, and the sum of all the digits is 20; if the digits be reversed, the new number is 1089 greater than the original number. Find the number. 3674 6347

3. A, B, and C subscribed $80 to a certain fund; if A's subscription had been 10 per cent less and B's 10 per cent more, C's must have been increased $1.60 to make up the sum; but if A's had been $12\frac{1}{2}$ per cent more, and B's $12\frac{1}{2}$ per cent less, C's subscription would have been $14. What sum did each subscribe? A $40 B $24 C $16

4. Two bodies are d yards apart. When they move towards each other with uniform velocities, they meet in m seconds; when one moves to overtake the other, they come together in n seconds. Find the rate of each body per second.

$d(m+n) \div 2mn$ $d(n-m) \div 2mn$

5. Find two numbers whose sum, difference, and product are as 5 : 1 : 18. 9, 6

6. What A is worth added to l times what B and C are worth is equal to p; what B is worth added to m times what A and C are worth is equal to q; what C is worth added to n times what A and B are worth is equal to r. Find what each is worth. $x = \frac{l(m-1)(q-mr) - (1-mn)mp - [illegible]}{m^2 l(1-n) + mnl(1-m) - [illegible](1- [illegible]}$

$n(\ -m)mr-q$ [illegible]

7. What number consisting of three digits is greater by 99 than when its digits are inverted, greater by 126 than the sum of its digits, and greater by 27 than when its tens and units are transposed? 130

8. A train (A) starts from P towards Q, and an hour afterwards another train (B), going ten miles an hour faster, starts from Q towards P, and they meet half way between P and Q; had A started an hour after B, its rate must have been increased 28 miles an hour in order to meet B at the same point. Find the length of the line from P to Q. 315 miles

9. In a mile race A gives B a start of 100 yards and beats him by 15 seconds; in running the same distance again, A gives B a start of 45 seconds and is beaten by 22 yards. Find the rate of each in miles per hour. 9, $8\frac{2}{11}$, miles per hr

10. The sum of three numbers is $(p+1)(q+1)n$; the sum of the two larger is equal to p times the smallest, and the sum of the two smaller, to q times the largest. Find the numbers.

11. Find three numbers such that the product of the first and second is equal to the sum of the same; the product of the first and third is equal to twice the sum of the same; and the product of the second and third is equal to three times the sum of the same.

12. I row a distance of c miles down stream in a hours, and back in b hours; find my rate of rowing in still water, and the velocity of the stream.

13. A traveller making p miles in q hours, is followed at an interval of m hours by one making r miles in s hours; in how many hours will the second overtake the first?

14. A pays to B and C as much as each of them has; B pays to A and C half as much as each of them has; and C pays to B and A a third of what each has; A has then \$1.44, B \$2.08, and C 96 cents. How much had each at first?

15. A person walks from A to B, a distance of $9\frac{1}{4}$ miles, in 2 hours and 52 minutes, and returns in two hours and 44 minutes. His rate of walking up hill, down hill, and on the level is 3, $3\frac{3}{4}$, and $3\frac{1}{4}$ miles, respectively. Find the length of level ground between A and B.

16. Two men, A and B, run at uniform paces from one station to another 4000 feet off. A starts 30 seconds after B, and arrives at the second station 10 seconds before B. Where does A pass B?

17. Seventeen gold coins, all of equal value, and as many silver coins, are placed in a row at random; A is to have one-half of the row, B the other half; A's share is found to include 7 gold coins, and the value of it is \$24; the value of B's share is \$27. Find the value of each gold and silver coin.

18. The road from A to D passes through B and C successively. The distance between A and B is 6 miles greater than between C and D; the distance between A and C is $\frac{1}{16}$ of a mile short of being half as great again as that between B and D, and the point half way from A to D is between B and C half a mile from B. Determine the distance between A and B, B and C, and C and D.

19. A laid out money in shares of two companies; if he had sold out 6 months later he would have lost \$100 less than half his outlay in the shares of the first company, $\$46\frac{2}{3}$ more than $\frac{1}{3}$ of his outlay in those of the second, and on the whole would have received \$40 less than $\frac{5}{8}$ of his whole original outlay. When he did sell out he lost \$40 less than $\frac{1}{5}$ of his outlay in the first company, and gained \$56 less than $\frac{1}{10}$ of his outlay in the second, so that his loss on the

whole, together with $\frac{1}{20}$ of his whole original outlay, made \$376. Find the sums laid out in the shares of each company.

20. To complete a certain work A requires m times as long a time as B and C together; B requires n times as long a time as A and C together; and C requires p times as long a time as A and B together. Show that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

21. A stage-coach left B for A two hours after a locomotive left A for B and they met in 6 hours after the locomotive started; had the speed of each been $1\frac{3}{4}$ miles an hour greater, they would have met in $5\frac{1}{2}$ hours; had the speed of each been $1\frac{3}{4}$ miles an hour less and had the stage-coach left two hours later than the locomotive, they would have met 7 hours and 5 minutes after the departure of the locomotive. Find the actual rate per hour of each, and the distance between A and B.

22. Separate the quotient $\dfrac{27+34z}{(3+4z)(6+7z)}$ into the sum of two quotients whose divisors are, respectively, $3+4z$ and $6+7z$.

23. Find two fractions whose denominators are $c-dz$ and $e-fz$, respectively, and whose sum is $\dfrac{a-bz}{(c-dz)(e-fz)}$.

24. There are three fractions whose denominators are, respectively, $8x-7$, $5x-4$, and $2x-1$, and whose sum is $\dfrac{306x^2-450x+162}{(8x-7)(5x-4)(2x-1)}$. Find them.

25. A train carries first, second, and third class passengers a certain distance at the respective fares of 84 cents, 60 cents, and 40 cents. The number of first and third class passengers is two less than the number of second-class passengers; the total fares paid by the first-class passengers was \$1.44 more than those paid by second-class; and the first and second class fares together are 24 cents more than $\frac{8}{3}$ of the third-class fares. Find the number of passengers of each class.

CHAPTER XIV.

QUADRATIC EQUATIONS.

154. A Quadratic Equation, or **Equation of the Second Degree,** is one which contains the *second, but no higher power*, of the unknown quantity. If an equation contains *only the second* power of the unknown quantity it is called a **pure quadratic.** If it contains *both the first and the second* power it is called an **adfected quadratic.** Thus $x^2 - 16 = 0$, $ax^2 - b = 0$ are pure quadratics; and $x^2 - 5x - 6 = 0$, and $ax^2 + bx + c = 0$ are adfected quadratics.

If an equation contains the *third power* of the unknown quantity it is called a **cubic equation**, or an equation of the *third degree;* if the fourth power is involved, the equation is of the *fourth degree*, and so on.

155. Roots of Equations.—We have seen that a simple equation can be reduced to the form $ax + b = 0$, and that the value of x which *satisfies* the equation, that is, which reduces $ax + b$ *to zero*, is the *root* of the equation; thus, putting $-\frac{b}{a}$ for x, we find $a \cdot \left(-\frac{b}{a}\right) + b$ to be zero, so that the equation is reduced to $0 = 0$, *i. e., it is satisfied.*

In the equation $2x - 6 = 0$, the *root* is 3, for $2 \cdot 3 - 6$ is zero. Now, in the equation $2x^2 - 6x = 0$, what value, or values, of x will make $2x^2 - 6x$ become zero? The expression has *two* factors, thus, $x(2x - 6)$, and therefore vanishes (becomes zero) when *either factor vanishes*, *i. e.*, when either $x = 0$, or $2x - 6 = 0$, which gives $x = 3$: from this it appears that the equation has *two roots*, viz. $x = 0$, and $x = 3$.

In the equation $x^2 - 16 = 0$, we have, by factoring,

$$(x + 4)(x - 4) = 0;$$

and the equation is satisfied if either

$x + 4 = 0$, *i. e.* if $x = -4$,

or $x - 4 = 0$, *i. e.* if $x = 4$.

The equation $x^3 - 16x = 0$, becomes, by factoring, $x(x^2 - 16) = 0$, *i. e.*, $x(x+4)(x-4) = 0$, and if any one of the *three* factors vanishes, the equation is satisfied; that is if $x = 0$, or $x + 4 = 0$, or $x - 4 = 0$; so that 0, -4, and 4 are the *roots* of the equation. The equation $x^3 - 4x^2 - 17x + 60 = 0$, is satisfied for $x - 3 = 0$, *i. e.* for $x = 3$; dividing by $x - 3$ we get the quadratic factor $x^2 - x - 20$, which has the factors $x + 4$, $x - 5$; hence the given equation may be put in the form $(x-3)(x+4)(x-5) = 0$, and is satisfied if any one of the *three* factors vanishes; *i. e.*, if $x - 3 = 0$, or $x + 4 = 0$, or $x - 5 = 0$; so that the *roots* are 3, -4, and 5.

The equation $x^4 - 13x^2 + 36 = 0$ can be put in the form $(x^2 - 9)(x^2 - 4) = 0$, which is satisfied if $x^2 - 9 = 0$, or $x^2 - 4 = 0$,

$$i.\ e., \text{ if } (x+3)(x-3) = 0, \text{ or } x = -3, \text{ or } +3;$$

$$\text{and if } (x+2)(x-2) = 0, \text{ or } x = -2, \text{ or } +2.$$

So that the *four roots* of the equation are $+3$, -3, $+2$, -2.

156. From these examples we infer :—

1°. That a **root of an equation** of any degree in x is any value of x which reduces the equation to the ultimate form $0 = 0$.

2°. That if any factor, or factors, of the form, $x - a$ can be found (where a may be zero), then each of these factors equated to zero (*i. e.*, $x - a = 0$, etc.) will give *one* root of the equation.

3°. That, conversely, if we find that any expression, such as $x - a$, etc., when put = zero, reduces the equation to the form $0 = 0$, then $x - a$, etc. $= 0$, *i. e.*, $x = a$, or a is *one root* of the equation.

4°. That the *degree* of an equation determines the number of its roots, *i. e.*,

an equation of the first degree (simple equation) has *one* root,
" " " " second " (quadratic) " *two* "
" " " " third " (cubic) " *three* "
etc., etc.

5°. That by multiplying together factors of the form $x - a$, $x - b$, $x - c$, etc., and equating the product to zero, we get an equation whose *roots* are a, b, c, etc.

6°. That, conversely, in *solving* an equation we are seeking the *linear* (*i. e.*, *one-dimension*) *factors* into which it may be resolved.

157. Pure Quadratics. — A quadratic equation may be considered as the product of *two* binomial factors equated to zero. Thus $(x + a)(x + b) = 0$. If $b = -a$, this becomes.

$$x^2 - (a - a)x - a^2 = 0, \ i.\ e.,\ x^2 - a^2 = 0,$$

which may be taken as the general form of a *pure quadratic.*

An equation of this form may be solved :

1°. By factoring : $(x + a)(x - a) = 0$;

$$\therefore x + a = 0, \text{ or } x = -a \text{ ; and } x - a = 0, \text{ or } x = a.$$

2°. By transposing, and extracting the square root of both sides : thus, $x^2 = a^2$; $\therefore x = +a$, or $-a$, since every quantity *has two square* roots. The two roots are usually written $\pm a$.

Ex. 1. $7x^2 - 63 = 0$.

Here divide by 7 ; $\therefore x^2 - 9 = 0$. Hence,

$$1°.\ (x + 3)(x - 3) = 0 \text{ ; } \therefore x = \pm 3,$$

or, $$2°.\quad x^2 = 9 \text{ ; } \therefore x = \pm 3.$$

Ex. 2. $11 - \dfrac{x + 25}{x^2} = 3 - \dfrac{x - 25}{x^2}$.

Clear of fractions, $\therefore 11x^2 - x - 25 = 3x^2 - x + 25$;

transpose, $$\therefore\ 8x^2 = 50,$$

and $$x^2 = \frac{50}{8} = \frac{25}{4} \text{ ;}$$

$$\therefore\ x = \pm \frac{5}{2}.$$

Ex. 3. $10{,}000 - \frac{36}{49}x^2 = 199$; transpose, and change signs to make the term in x^2 *positive*, then

$$\tfrac{36}{49}x^2 = 10{,}000 - 199 = 9801 \text{ ;}$$

take square root of both sides, $\therefore \frac{6}{7}x = \pm 99$,

and $$x = \pm 115\tfrac{1}{2}.$$

In this case, since the coefficient of x^2 is a perfect square, we extract the square root of both sides before dividing.

Ex. 4. $\dfrac{x+a}{x-a} + \dfrac{x-a}{x+a} = \dfrac{2(a^2+1)}{(1+a)(1-a)}$.

Reduce, $$\therefore \frac{(x+a)^2 + (x-a)^2}{x^2 - a^2} = \frac{2(a^2+1)}{1-a^2};$$

$$\therefore \frac{(x^2+a^2)}{x^2-a^2} = \frac{a^2+1}{1-a^2},$$

now clear of fractions,

$$\therefore (1-a^2)(x^2+a^2) = (a^2+1)(x^2-a^2);$$

transpose, and collect coefficients of x^2,

$$\therefore \{(1-a^2) - (a^2+1)\}x^2 = -(1-a^2)a^2 - (1+a^2)a^2,$$

or $$-2a^2x^2 = -2a^2;$$

$$\therefore x^2 = 1, \text{ and } x = \pm 1.$$

Ex. 5. Form the equation whose roots are 5 and -5;

$$(x+5)(x-5) = 0,$$

or $$x^2 - 25 = 0;$$

for the left side vanishes if

$$x + 5 = 0, \textit{ i. e. } \text{ if } x = -5,$$

and if $$x - 5 = 0, \textit{ i. e. } \text{ `` } x = +5.$$

Ex. 6. Form the equation whose roots are $\frac{3}{4}$ and $-\frac{3}{4}$, we have

$$\left(x - \frac{3}{4}\right)\left(x + \frac{3}{4}\right) = 0,$$

or $$x^2 - \frac{9}{16} = 0, \text{ or } 16x^2 - 9 = 0.$$

Ex. 7. What is the equation whose roots are $\pm(a+b)$?

We have $$\{x + (a+b)\}\{x - (a+b)\} = 0,$$

$$\textit{i. e. } x^2 - (a+b)^2 = 0.$$

EXERCISE LXXVI.

Solve the equations in the following examples :—

1. $x^2 - 9 = 0$: $x^2 - 36 = 0$; $x^2 = 81$; $x^2 - a^2 = 0$; $x^2 - \frac{1}{4}b^2 = 0$.

2. $16x^2 - 25 = 0$: $x^2 - \frac{1}{16} = 0$; $5x^2 = 125$; $7x^3 - 112x = 0$.

3. $x^2 = 1$; $x^2 - \frac{1}{4} = 0$; $5x^2 - \frac{1}{5} = 0$; $\frac{x}{4} = \frac{9}{x}$; $x^2 = 12$.

4. $x^2 - m = 0$; $x^2 - 15 = 0$; $(x - 1)^2 = 4$; $(x - a)^2 = b^2$.

$x = 3, -1$ $x = a \pm b$

5. $(x - 1)^2 = 9$; $(x - 5)^2 = 7$; $\left(x - \frac{1}{2}\right)^2 = \frac{1}{4}$; $\left(x - \frac{2}{3}\right)^2 - \frac{25}{9} = 0$.

$4, -2$ $5 \pm \sqrt{7}$ $1, 0$ $\frac{7}{3}$ or -1

6. $4(x+3)^2 = (x-4)^2$; $x^2 = (a-b)^2$; $x^2 - (1-a)^2 = 0$; $\left(x - \frac{3}{4}\right)^2 = a^2$.

-10 or $-\frac{2}{3}$ $x = \pm(a-b)$ $\pm(1-a)$ $\frac{3}{4} \pm a$

7. $x^2 - (3a-2b)^2 = 0$; $(a-b)^2x^2 = a^2 + 2ab + b^2$; $\frac{1}{4}(x-1)^2 = (x-2)^2$.

$\pm(3a-2b)$ $\pm(a+b) \div (a-b)$ 3 or $\frac{5}{3}$

8. $(x+a-b)^2 - (a+b)^2 = 0$; $7x^2 - 105{,}903 = 0$; $16x^2 - 1{,}210{,}000 = 0$.

$-2b, -2a$ ± 123 ± 275

9. Form the equation whose roots are ± 3 ; ± 9 ; $\pm \frac{1}{2}$; $\pm ab$; $\pm \frac{1}{2}a$; $\pm(1 + a)$; $\pm(a - 1)$.

10. $\frac{x}{2} - \frac{x-2}{x+4} = 1$; $\frac{4x}{5} + \frac{x-5}{x} = 1$; $\frac{9x}{7} + \frac{x-7}{x} = 1$.

± 2 $\pm \frac{5}{2}$ $\pm \frac{7}{3}$

11. Form equations with roots $\pm \frac{1}{2}(a \div b)$; $\pm \frac{a}{b}$;

× $\pm\left(\frac{1}{a} - \frac{1}{b}\right)$; $\pm\left(\frac{a}{b} + \frac{b}{a}\right)$; $\pm\sqrt{5}$; $\pm\sqrt{ab}$.

12. $\frac{x}{6} + \frac{x + \frac{1}{3}}{x + 3} = \frac{3}{2}$; $\frac{x+a}{x-a} + \frac{x-a}{x+a} = 4\frac{1}{4}$; $\frac{x}{a} + \frac{x-a}{x} = 1$.

± 5 $\pm \frac{5a}{3}$ $\pm a$

13. $\frac{x-m}{x+m} = \frac{n-x}{n+x}$; $\frac{49}{64}\left(x - \frac{7}{9}\right)^2 = \frac{25}{81}$; $\frac{(a-x)(x-b)}{a+b-x} = x$.

$\pm\sqrt{mn}$ $\frac{89}{63}, \frac{1}{7}$ α

14. Form the equations whose roots are $\pm(2a - 3b)$; $\pm\sqrt{a^2 + b^2}$; $\pm \frac{a+b}{ab}$; $\pm \frac{1-a}{1+a}$; $\pm \frac{a+b}{a-b}$.

15. $\frac{2}{x-10} + 10 - x = \frac{2}{10-x}$; $\frac{x+a}{x-1} + \frac{x-a}{x+1} = 2a$.

$12, 8$ $\pm\sqrt{\frac{a+1}{a-1}}$

16. $\dfrac{ab}{a^2-b^2x^2}+\dfrac{cd}{c^2-d^2x^2}=0$; $\dfrac{a+x}{a-x}=\dfrac{x+b}{x-b}$; $\dfrac{a-x}{1-ax}=\dfrac{1-bx}{b-x}$.

$\pm\sqrt{\frac{ac}{bd}}$ $\pm\sqrt{ab}$ ± 1

17. $\dfrac{a-x}{1-ax}=\dfrac{b-x}{1-bx}$; $\dfrac{a(a-b)}{x-a-b}+a+b-x=\dfrac{(b-a)b}{a+b-x}$.

± 1 $2a, 0$

18. $\dfrac{x+m-2n}{x+m+2n}=\dfrac{n+2m-2x}{n-2m+2x}$; $\dfrac{x+a+2b}{x+a-2b}=\dfrac{b-2a+2x}{b+2a-2x}$.

$\pm\sqrt{m^2+n^2}$ $\pm\sqrt{a^2+b^2}$

19. $\dfrac{5a-6b+x}{a+x}=\dfrac{3a-5b+3x}{a+b+x}$; $\dfrac{a+b-x}{3a-b-3x}=\dfrac{3(a-b+x)}{a-5b+x}$.

$\pm\sqrt{a^2+2ab-3b^2}$ $\pm\sqrt{a^2+ab+b^2}$

20. $\dfrac{x+m}{x-m}+\dfrac{x-m}{x+m}=\dfrac{2(m^2+1)}{(1+m)(1-m)}$. ± 1

158. Problems in Pure Quadratics.—A few examples are here given of *problems which are solved by pure quadratics.*

Ex. 1. If I multiply by itself the number of dollars which I have, I shall have \$132¼. How much have I?

Let x = No. of dollars; then $x \cdot x$, *i. e.* $x^2 = 132\frac{1}{4}$;

or $x^2 = \frac{529}{4}$; $\therefore\ x = \pm\frac{23}{2} = \pm 11\frac{1}{2}$,

where $+11\frac{1}{2}$ must be taken.

Ex. 2. Find two numbers in the ratio of 11 : 13, whose product shall be 7007.

Let $11x$ and $13x$ be the numbers; then $11x \cdot 13x$ is their product.

$$\therefore\ 11 \cdot 13 \cdot x^2 = 7007; \qquad \therefore\ x^2 = 49 \text{ and } x = \pm 7.$$

$$11 \times (\pm 7) = \pm 77. \text{ and } 13 \times (\pm 7) = \pm 91;$$

$\therefore$ the numbers are ± 77, and ± 91.

Ex. 3. The length of a rectangle is to its breadth as 3 : 4, and its area is 1200 square yards; find its dimensions.

Let $4x$ and $3x$ represent the length and breadth, respectively.

Then $4x \times 3x = 12x^2 = 1200$; $\therefore x^2 = 100$; and $x = \pm 10$;

$\therefore$ the sides are 40 and 30.

Ex. 4. In a concert-room 800 persons are seated on benches of equal length; if there were 20 fewer benches, it would be necessary for two persons more to sit on each bench; find the number of benches.

Suppose $x-1$ persons sit on each bench in the first case;

then $x+1$ " " " second case.

There are $\frac{800}{x-1}$ benches in the first case; and $\frac{800}{x+1}$ in the second case;

$$\therefore\ \frac{800}{x-1}-\frac{800}{x+1}=20, \text{ or } 40\left(\frac{1}{x-1}-\frac{1}{x+1}\right)=1, \text{ or } \frac{80}{x^2-1}=1;$$

$$\therefore\ x^2=81, \quad \text{and} \quad x=\pm 9.$$

EXERCISE LXXVII.

PROBLEMS.

1. If the number of dollars I have in my pocket were multiplied by itself, I should have $132\frac{1}{4}$. How much have I? $\$\frac{23}{2}$
2. Find a number such that its fifth part multiplied by its seventh part is equal to 4235. 385
3. If I multiply $3\frac{4}{7}$ times a given number by 8.68 times the same number, the result is 5239. Find the number. 160 3
4. Find two numbers in the ratio of 11 : 13, of which the product is 9152. 88 104
5. I multiply the third part of a certain number by its fourth part, and the product by the fifth part of the number; and find the result to be the sixth part of the number. Find the number. $\sqrt{10}$
6. Find three numbers in the proportion of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, such that the sum of their squares shall be 10309. 78, 52, 39
7. A rectangular field which has a length of 3367 feet, and a breadth of 37 yards, has the same number of square yards as a rectangular field of which the length and breadth are as 13 : 7. Find the dimensions of the latter. 481 √3, 71

8. Find three equal numbers whose sum is equal to their product. $\pm\sqrt{3}$, 0

9. A man bought a number of pounds of raisins, twice as much coffee, and four times as much tea; he paid for every pound of each article 10 times as many cents as there were pounds of the article, and his outlay was $8.40. How many pounds of each did he buy? 2, 4, 8

10. A certain number is increased by 3, and also diminished by 3; the greater sum is then divided by the less, and the less, by the greater; the sum of the quotients is $3\frac{1}{4}$. Find the number. $\pm\sqrt{\frac{189}{5}}$

11. A man was instructed to buy 18 yards of cotton if it cost just 18 cents a yard; but if it was cheaper or dearer, he was to buy as many yards *more* or *less* as the price was below or above 18 cents; he expended $3.15. How many yards did he buy? 15 or 21

12. What is the result in the last question, if a is put for 18 and b for $3.15? $a \pm \sqrt{a^2 - b}$

13. A string of a certain length will just go round a square; if the string is shortened 8 ft. it will just go round another square, whose area is $\frac{16}{25}$ of that of the first square. Find the length of the string. 40 ft

14. Divide 20 into two such parts that the squares of the parts shall have the ratio $1 : 2\frac{1}{4}$. 12, 8

15. If the side of a square be increased 3 feet, its area will be increased by $\frac{9}{16}$ of itself. Find the side. 12 ft

16. Two farmers sold in all 260 eggs at market, and each realized the same amount; the first said to the second: "If I had had *your* eggs and sold them at *my* price, I should have got $7.20 for them." The second said: "But if *I* had had *your* eggs and sold them at *my* price, I should have got $9.80 for them. How many eggs did each take to market? 140 + 120

17. A man sold 133 yards of cloth at a certain per cent profit: he invested the proceeds in goods on which he realized the same profit; with the proceeds of the last sale he was able

to buy 168 yards of cloth which cost 14 per cent more per ct than he paid for the 133 yards: what per cent profit had he on the 133 yards? 20 %

18. One side of a right-angled triangle is $3\frac{3}{7}$ times the other, the hypothenuse being 1000 feet. Find the sides. 960, 280

19. Two sides of a rectangle are in the ratio of 15 : 8; the diagonal is 323 feet. Find the sides. 152 + 285

20. Two persons start from the same place, the one travelling due east at the rate of 6 miles an hour, the other, due north at $4\frac{1}{2}$ miles an hour; in how many hours will they be 30 miles apart? 4 hrs

159. Adfected Quadratics.—In many cases, the factors which form an adfected quadratic may be found, and its roots at once determined, by some of the methods of factoring already given.

Ex. 1. $$x^2 - 5x + 6 = 0.$$

This can at once be put in the form

$$(x - 3)(x - 2) = 0;$$

$$\therefore \quad x - 3 = 0, \quad \text{or} \quad x - 2 = 0,$$

and $$x = 3 \text{ or } 2.$$

Ex. 2. $$x^2 + x - 20 = 0.$$

By factoring we have $(x + 5)(x - 4) = 0$;

$$\therefore \quad x + 5 = 0, \quad \text{or} \quad x - 4 = 0,$$

and $$x = -5 \text{ or } 4.$$

Ex. 3. $$6x^2 - x - 12 = 0.$$

By factoring we have

$$(2x - 3)(3x + 4) = 0;$$

$$\therefore \quad 2x - 3 = 0, \quad \text{or} \quad 3x + 4 = 0,$$

and $$x = -\tfrac{4}{3} \text{ or } \tfrac{3}{2}.$$

Ex. 4. $$x^3 + 2x^2 - 35x = 0.$$

Here we have $$x(x + 7)(x - 5) = 0;$$

$$\therefore\quad x = 0, \quad\text{or}\quad x + 7 = 0, \quad\text{or}\quad x - 5 = 0,$$

and $$x = 0, \text{ or } -7, \text{ or } 5.$$

Ex. 5. $$x^2 + 3\tfrac{1}{2}ax - 2a^2 = 0.$$

We have at once $$(x - \tfrac{1}{2}a)(x + 4a) = 0;$$

$$\therefore\quad x - \tfrac{1}{2}a = 0, \quad\text{or}\quad x + 4a = 0,$$

and $$x = \tfrac{1}{2}a, \text{ or } -4a.$$

Ex. 6. $$x^2 - 3x + ax - 3a = 0.$$

By factoring we have $(x - 3)(x + a) = 0$;

$$\therefore\quad x - 3 = 0, \quad\text{or}\quad x + a = 0, \text{ etc.}$$

Ex. 7. $$x^3 - 4x = 8x + 16.$$

By factoring we have $x(x^2 - 4) = 8(x + 2)$;

where $x + 2$ is a *factor* of each side;

$$\therefore\quad x + 2 = 0.$$

Dividing by $x + 2$, we get

$$x(x - 2) = 8;$$

$$\therefore x^2 - 2x - 8 = 0;$$

from which we have

$$(x - 4)(x + 2) = 0.$$

Hence the given equation is satisfied if

$$x + 2 = 0, \quad\text{or}\quad x - 4 = 0, \quad\text{or}\quad x + 2 = 0;$$

$$\therefore\quad x = -2,\ 4,\ -2.$$

Ex. 8. $$x^3 + 7x = 49(x + 1).$$

Here we can arrange the terms thus :—

$$x^3 - 49x = -7x + 49,$$

or $$x(x^2 - 49) = -7(x - 7),$$

where $x - 7$ is a factor of each side;

$$\therefore\quad x - 7 = 0, \quad\text{or}\quad x = 7.$$

Dividing by $x - 7$, we get

$$x^2 + 7x + 7 = 0,$$

of which the solution may be found by the method of Art. 161.

160. Conversely, *a quadratic may be formed for any given roots.*

Ex. 1. Form the quadratic whose roots shall be -7 and -3.

We have at once $\{x - (-7)\}\{x - (-3)\} = 0$;

$$\therefore \quad (x + 7)(x + 3) = 0,$$

or
$$x^2 + 10x + 35 = 0,$$

of which the roots are -7 and -3.

Ex. 2. Form the equation which shall have roots -9 and $+8$. We have

$$(x + 9)(x - 8) = 0;$$

$$i.\,e.,\ x^2 + x - 72 = 0,$$

whose roots are -9 and $+8$.

Ex. 3. Form the equation whose roots are -4 and $5a$. We have

$$(x + 4)(x - 5a) = 0,$$

or
$$x^2 + (4 - 5a)x - 20a = 0.$$

Ex. 4. Form an equation whose roots are a and $-b$. We have

$$(x - a)(x + b) = 0,$$

or
$$x^2 - (a - b)x - ab = 0.$$

Ex. 5. Form an equation with roots $a - 1$ and $a + 1$. As before,

$$(x - \overline{a - 1})(x - \overline{a + 1}) = 0;$$

$$i.\,e.,\ x^2 - 2ax + a^2 - 1 = 0.$$

Ex. 6. What is the equation whose roots are $\frac{a}{b}$ and $-\frac{b}{a}$? We have

$$\left(x - \frac{a}{b}\right)\left(x + \frac{b}{a}\right) = 0;$$

$$i.e., \; x^2 - \left(\frac{a}{b} - \frac{b}{a}\right)x - 1 = 0.$$

EXERCISE LXXVIII.

Solve the following equations :—

1. $(x - 3)(x - 4) = 0$; $(x - 4)(x + 5) = 0$; $x(x - 5) = 0$.
2. $(x - 3a)(x + 4a) = 0$; $x(x - a) = 0$; $(x - a)(x - b) = 0$.
3. $2x(x - 7) = 0$; $x^2 - 15x = 0$; $ax^2 = 3ax$; $x(x + b) = 0$.
4. $3x^2 - 13x = 0$; $(2x - 1)(x - 3) = 0$; $(2x - 5)(3x + 4) = 0$.
5. $x^2 - 3x + 2 = 0$; $x^2 - 7x + 12 = 0$; $x^2 + 9x + 20 = 0$.

   $x = 1, 2$ $x = 3, 4$ $x = -5, -4$
6. $(x - a)(x - bc) = 0$; $x^2 - 8x + 15 = 0$; $x^2 + 8x - 33 = 0$.

   $x = 3, 5$ $x = 3, -11$
7. $x^2 - 17x + 60 = 0$; $x^2 - 17x - 60 = 0$; $4x^2 - 2x - 20 = 0$.

   $x = 5, 12$ $x = 20, -3$ $x = \frac{5}{2}, -2$
8. $x^2 - (a + b)x + ab = 0$; $x^2 - (1 - a)x - a = 0$; $x^2 - 1\frac{1}{2}x - 1 = 0$.

   $x = a, b$ $x = 1, -a$ $x = 2, -\frac{1}{2}$

Form the equations in the following cases :—

1. With roots 3 and 4; 4 and 6; − 3 and − 4; − 7 and 9; 8 and − 5; 13 and − 10. $(x-3)(x-4)=0$; $(x-4)(x-6)=0$; $(x+3)(x+4)=$
   $(x+7)(x-9)=0$ $(x-8)(x+5)=0$; $(x-13)(x+10)=0$
2. With roots $-a$ and $+2a$; $-3a$ and $5a$; 1 and $-a$.
   $(x+a)(x-2a)=0$; $(x+3a)(x-5a)=0$; $(x-1)(x+a)=0$
3. With roots $2b$ and $2a$; a and $-b$; $-a$ and $-b$; 3 and b.
   $(x-2b)(x-2a)=0$; $(x-a)(x+b)=0$; $(x+a)(x+b)=0$; $(x-3)(x-b)=$
4. With roots $\frac{1}{2}$ and 2; $\frac{1}{2}$ and $-\frac{3}{4}$; $2a$ and $\frac{1}{2}$; a^2 and $-b^2$.
   $(x-\frac{1}{2})(x-2)=0$; $(x-\frac{1}{2})(x+\frac{3}{4})=0$; $(x-2a)(x-\frac{1}{2})=0$; $(x-a^2)(x+b^2)=$

What inferences are to be drawn from the following statements?

1. $x \cdot y = 0$.
2. $x \cdot y \cdot z = 0$.
3. $4x = 0$.
4. $5xy = 0$.
5. $axyz = 0$.
6. $x(a + b) = 0$.
7. $c(x - a) = 0$.
8. $(x - a)(b + c) = 0$.
9. $x(1 - a)(1 - b) = 0$.
10. $(x - ac)(x - bc) = 0$.

161. Completing the Square.—The factors, and therefore the *roots* of any quadratic may be determined by the method of *completing the square.*

If all the terms containing x, when collected on one side of the equation, form a complete square, the equation may be solved by extracting the square root of both sides as in the case of a *pure quadratic.* Thus :—

Ex. 1. $x^2 - 2x + 1 = 5$; the left hand side is a complete square.

$$\therefore x - 1 = \pm\sqrt{5}, \text{ and } x = 1 \pm \sqrt{5}.$$

Or, thus : $(x - 1)^2 - 5 = 0$; $\therefore (x - 1 + \sqrt{5})(x - 1 - \sqrt{5}) = 0$;

whence $x = 1 - \sqrt{5}$, or $1 + \sqrt{5}$.

Ex. 2. $x^2 - 2ax + a^2 = b$. Here the left side is a square,

$$\therefore x - a = \pm\sqrt{b}, \text{ and } x = a \pm \sqrt{b}.$$

If the x-terms *do not form a square* we can, in every case, *complete the square* by adding the necessary quantity to both sides of the equation. To determine this quantity, we have only to consider how the square of a binomial is formed. Take the *binomial* $x + a$, whose *first term* is x, and *second* term a.

Then $(x + a)^2 = x^2 + 2a \cdot x + a^2 \ldots$, the *trinomial product.*

Now, if only the *two* terms, $x^2 + 2a \cdot x$, of this are given, how can we determine a^2, the third term?

Observe that,

1°. The third term of the product is the square of a (the second term of the *binomial*).

2°. The coefficient of x is *always* **twice** a.

3°. And $\therefore a$ is **half** the coefficient of x.

Hence, **to complete the square** in any case, *add the square of half the coefficient of x* (the unknown quantity) *to both sides of the equation.*

N. B.—If the equation is of the form $X = 0$, *i. e.*, if *all the terms are on one side*, we may *add* and *subtract* the square of half the coefficient of x, and factor the result.

Ex. 1. $10x - x^2 = 21$. Here we first change the signs of all the terms, so that x^2 *may be positive*, thus :—

$$x^2 - 10x = -21$$

half the coefficient of x is 5 ;

$$\therefore x^2 - 10x + 25 = -21 + 25, \quad = 4$$

$$\therefore x - 5 = \pm 2, \text{ and } x = 5 \pm 2 = 7, \text{ or } 3.$$

Or we may proceed thus :—

$$x^2 - 10x + 21 = 0 ;$$

Add the pair of complementary terms $5^2 - 5^2$,

$$\therefore x^2 - 10x + 5^2 - 5^2 + 21 = 0,$$

or $$(x - 5)^2 - 4 = 0,$$

$$\therefore (x - 5 + 2)(x - 5 - 2) = 0,$$

or $(x - 3)(x - 7) = 0$, whence $x = 3$, or 7.

Ex. 2. $3x^2 - 21x - 450 = 6000$.

Divide by 3 and transpose,

$$\therefore x^2 - 7x = 2150 ;$$

Add $\left(\frac{7}{2}\right)^2$ to both sides,

$$\therefore x^2 - 7x + \left(\frac{7}{2}\right)^2 = 2150 + \left(\frac{7}{2}\right)^2 = \frac{8649}{4}.$$

$$\therefore x - \frac{7}{2} = \pm \frac{93}{2} ;$$

and $$x = \frac{7}{2} \pm \frac{93}{2} = 50, \text{ or } -43.$$

N. B.—By factors : $x^2 - 7x - 2150 = 0$,

or $(x - 50)(x + 43) = 0$,

$\therefore x$, as before, $= 50$, or -43.

Ex. 3. $\frac{x+3}{x+2}+\frac{x-3}{x-2}=\frac{2x-3}{x-1}$.

First complete the divisions,

$$\therefore 1+\frac{1}{x+2}+1-\frac{1}{x-2}=2-\frac{1}{x-1},$$

$$\therefore \frac{1}{x+2}-\frac{1}{x-2}=-\frac{1}{x-1},$$

$$\therefore \frac{-4}{x^2-4}=-\frac{1}{x-1},$$

or $\quad x^2-4x=0$; or $x(x-4)=0 \therefore x=0$, or 4.

Ex. 4. $x^2+169-\frac{156x}{5}+\frac{36x^2}{25}=74$.

Clear of fractions,

$$\therefore 25x^2+4225-780x+36x^2=1850;$$

Transpose, $\quad \therefore 61x^2-780x=-2375$,

$$x^2-\frac{780}{61}x+\left(\frac{390}{61}\right)^2=-\frac{2375}{61}+\left(\frac{390}{61}\right)^2=\frac{7225}{(61)^2},$$

$$\therefore x-\frac{390}{61}=\pm\frac{85}{61};$$

and $\quad x=\frac{390\pm 85}{61}=5$, or $\frac{475}{61}$.

Or, by factors, $61x^2-780x+2375=0=(x-5)(61x-475)$, etc.

EXERCISE LXXIX.

Complete the squares in the following cases :—

1. x^2-4x; x^2-10x; x^2+2ax; x^2-2ax.
x^2-4x+4 ; $x^2-10x+25$; $x^2+2ax+a^2$; $x^2-2ax+a^2$
2. x^2+12x; x^2-4cx; x^2-14x; x^2+3x; x^2+x.
$x^2+12x+36$; $x^2-4cx+4c^2$; $x^2-14x+49$; $x^2+3x+\frac{9}{4}$; x
3. x^2-9x; x^2+7x; $4x^2-12x$; $x^2-\frac{4}{3}x$; $x^2-\frac{5}{6}x$.
$x^2-9x+\frac{81}{4}$; $x^2+7x+\frac{49}{4}$; $x^2-3x+\frac{9}{4}$; $x^2-\frac{4}{3}x+\frac{4}{9}$; $x^2-\frac{5}{6}x+$
4. $x^2-\frac{3}{2}x$; $3x^2-4x$; $x^2-\frac{16}{5}x$; $x^2-\frac{7}{4}x$; x^2+ax.

$x^2-\frac{3}{2}x+\frac{9}{16}$; $x^2-\frac{4}{3}x+\frac{4}{9}$; $x^2-\frac{16}{5}x+\frac{64}{25}$; $x^2-\frac{7}{4}x+$

Solve the following equations :—

1. $x^2 + 6x = 7$; $x^2 - 8x = -12$; $x^2 + 10x = -21$.

$-7, 1$ $6, 2$ $-7, -3$

2. $x^2 + 3x + 2 = 0$; $x^2 + 2mx = n$; $x^2 - mx = n$.

$-2, -1$ $-m \pm \sqrt{m^2+n}$ $\frac{1}{2}\{m \pm \sqrt{m^2+4n}\}$

3. $x^2 - 10x - 24 = 0$; $x^2 + 26x + 120 = 0$; $26x - x^2 + 120 = 0$.

$12, -2$ $-6, -20$ $30, -4$

4. $22x + 23 - x^2 = 0$; $x^2 - \frac{7}{6}x - \frac{1}{2} = 0$; $\frac{19}{5}x = \frac{4}{5} - x^2$.

$23, -1$ $\frac{3}{2}, -\frac{1}{3}$ $\frac{1}{5}, -4$

5. $5x^2 - 17x + 12 = 0$; $2(x - \frac{1}{2}) - 4(x + \frac{1}{2})^2 = 134$. –

* $2\frac{2}{5}, 1$ $-\frac{1}{4} \pm \frac{1}{4}\sqrt{-543}$

6. $\frac{40}{x-5} + \frac{27}{x} = 13$; $\frac{x-1}{x-2} + \frac{1}{x} = 4\frac{7}{11}$; $\frac{x-2}{x-3} + \frac{x-4}{x-5} = 3$.

$\frac{15}{13}, 9$ $\frac{1}{40}\{51 \pm \sqrt{1721}\}$ $+5 \pm \sqrt{2}$

7. $\frac{x+3}{x-4} + \frac{x-2}{x+3} = 4\frac{26}{95}$; $\frac{x+9}{x+3} - \frac{x+2}{x-4} = 8\frac{24}{25}$.

$6\frac{1}{2}, -4\frac{67}{708}$ $3\frac{1}{4}, -2\frac{1}{4}$ –

8. $\frac{x+3}{2x-7} - \frac{2x-1}{x-3} = 0$; $\frac{1}{3-x} - \frac{4}{5} = \frac{1}{9-2x}$; $\frac{x}{100} - \frac{21}{25x} = \frac{1}{4}$.

1- $4, \frac{4}{3}$ $2, 4\frac{3}{8}$ $28, -3$

9. $\frac{9}{16} + \frac{64}{81x^2} = \frac{4}{3x}$; $\frac{2x+1}{2x+3} - \frac{3x+8}{3x-2} + \frac{13}{11} = 0$. $x = \frac{32}{27}$; $4, -1\frac{1}{6}$

10. $\frac{\frac{3}{2}x + 2}{\frac{3}{2}x - 1} - \frac{7}{3} = \frac{4 - \frac{3}{2}x}{3x}$; $\frac{x-7}{x-4} + \frac{2(x+9)}{x+5} = \frac{3x+13}{x+2}$. 1 or -23

2 or $-\frac{8}{15}$

11. $\frac{1}{4+x} + \frac{1}{x-3} + \frac{1}{2} = 0$; $\frac{x}{a+x} + \frac{a+x}{x} = 2\frac{1}{2}$. a or $-2a$

$x = \frac{-5 \pm \sqrt{65}}{2}$

12. $x^2 - (a^2 + b^2)x + (a^2 - b^2)ab = 0$; $(m - n)x^2 - nx = m$.

x $a(a-b)$ $b(a+b)$ -1 or $\frac{m}{m-n}$

13. $557x = 5{,}801\frac{1}{4} + 8x^2$; $\frac{x^2}{(m+n)^2} - \frac{4mn}{(m+n)^2}x = (m-n)^2$.

$x = 56\frac{7}{8}$; $12\frac{3}{4}$ $2mn \pm (m^2+n^2$

14. $(x - a)^2 - b(x - a - c) = bc$; $ax^2 - a^2(x + b^2) = ab(x - ab)$.

$x = a$ or $a + b$ $x = a + b$ or c

15. $\frac{x^2}{a+b} + a - b = \frac{2ax}{a+b}$; $\frac{1}{a} - \frac{1}{b} + \frac{2}{x} = \frac{2}{2(a-b)+x}$;   1 or -2

$a \pm b$

+ $\frac{x-a}{2x} - \frac{x-3}{x+a} = \frac{1}{2} - \frac{a}{6}$. $x = 3$, $a^2 \div (6-a)$

162. General Formula.—The general form of a quadratic equation is

$$ax^2 + bx + c = 0, \tag{1}$$

or by dividing by the coefficient of x,

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0; \qquad (2)$$

in which the square may be completed by adding the square of half of $\frac{b}{a}$, *i. e.*, the square of $\frac{b}{2a}$, to both sides of the equation.

The square may also be completed in the following manner :—

Multiply both sides of (1) by $4a$ (*i. e.*, by 4 *times* the coefficient of x^2) and add b^2 (*i. e.*, the square of the coefficient of x) to both sides; then

$$4a^2x^2 + 4abx + b^2 = b^2 - 4ac,$$

in which the left side is a square;

$$\therefore \quad 2ax + b = \pm \sqrt{(b^2 - 4ac)},$$

whence
$$x = \frac{-b \pm \sqrt{(b^2 - 4ac}}{2a}.$$

Or, by factoring, we may proceed thus :—

$$4a^2x^2 + 4abx + b^2 - b^2 + 4ac = 0,$$

or
$$(2ax + b)^2 - (b^2 - 4ac) = 0;$$

$$\therefore \quad \{2ax + b + \sqrt{(b^2 - 4ac)}\}\{2ax + b - \sqrt{(b^2 - 4ac}\} = 0;$$

hence,
$$2ax + b + \sqrt{(b^2 - 4ac)} = 0,$$

and
$$\therefore \quad x = \frac{-b - \sqrt{(b^2 - 4ac)}}{2a};$$

or
$$2ax + b - \sqrt{(b^2 - 4ac)} = 0,$$

and
$$\therefore \quad x = \frac{-b + \sqrt{(b^2 - 4ac)}}{2a};$$

or in one formula, as before,

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

N. B.—*The student should carefully remember this form, and acquire facility in writing down from it the roots of any quadratic equation.*

Ex. 1. $2x^2 - 7x + 3 = 0$.

Comparing this with the *general form*, we see that a is 2, b is -7, and c is 3, whence, applying the formula,

$$x = \frac{7 \pm \sqrt{(7^2 - 4\cdot2\cdot3)}}{2\cdot2} = 3, \text{ or } \frac{1}{2}.$$

N. B.—By factoring, we get $(2x-1)(x-3) = 0$; whence x is obtained, as by the formula.

Ex. 2. $\dfrac{2x+9}{9} + \dfrac{4x-3}{4x+3} = 3 + \dfrac{3x-16}{18}$.

Clearing of fractions, and transposing,

$$4x^2 + 139x + 72x - 216x = 114,$$

or

$$4x^2 - 5x - 114 = 0.$$

In applying the formula, we see that a is 4, b is -5, and c is -114;

$$\therefore \quad x = \frac{5 \pm \sqrt{(25 + 4\cdot4\cdot114)}}{2\cdot4}.$$

N. B.—By factoring we get $(4x+19)(x-6) = 0$; whence we get x, as by the formula.

163. *The artifices used* in diminishing labour, by *combining fractions*, etc., may be occasionally employed in solving quadratics, as in simple equations.

Ex. 1. $\dfrac{6x-12}{-x+5} + \dfrac{2x-11}{2x-5} = 7$.

Complete the divisions,

$$\therefore \quad -6 + \frac{18}{5-x} + 1 - \frac{6}{2x-5} = 7,$$

or

$$\frac{3}{5-x} - \frac{1}{2x-5} = 2;$$

that is,

$$4x^2 - 23x + 30 = 0.$$

Apply the formula, then

$$x = \frac{23 \pm \sqrt{(23^2 - 4 \cdot 4 \cdot 30)}}{2 \cdot 4} = 2 \text{ or } 3\tfrac{3}{4}.$$

Ex. 2. $\dfrac{6x+5}{2x-7} + \dfrac{4x-1}{x-2} = \dfrac{7x+1}{x-3}$.

$$\therefore \quad 3 + \frac{26}{2x-7} + 4 + \frac{7}{x-2} = 7 + \frac{22}{x-3},$$

or $$\frac{26}{2x-7} + \frac{7}{x-2} = \frac{22}{x-3};$$

whence $$\frac{40x - 101}{(x-2)(2x-7)} = \frac{22}{x-3};$$

$$\therefore \quad (x-3)(40x-101) - 22(x-2)(2x-7) = 0,$$

or $$4x^2 - 21x + 5 = 0,$$

that is $$(4x-1)(x-5) = 0;$$

whence $$x = 5 \text{ or } \frac{1}{4}.$$

Ex. 3. $\dfrac{1}{x+a+b} + \dfrac{1}{x-a+b} + \dfrac{1}{x+a-b} + \dfrac{1}{x-a-b} = 0$.

Take the first and the last fractions together, and also the two middle ones, thus

$$\left(\frac{1}{x+(a+b)} + \frac{1}{x-(a+b)}\right) + \left(\frac{1}{x-(a-b)} + \frac{1}{x+(a-b)}\right) = 0,$$

or $$\frac{x-(a+b)+x+(a+b)}{x^2-(a+b)^2} + \frac{x+(a-b)+x-(a-b)}{x^2-(a-b)^2} = 0;$$

which reduces to

$$\frac{2x}{x^2-(a+b)^2} + \frac{2x}{x^2-(a-b)^2} = 0,$$

where $2x$ is a factor, and gives $x = 0$ for *one* root. Divide by $2x$,

$$\therefore \quad x^2 - (a-b)^2 + x^2 - (a+b)^2 = 0,$$

or $$x^2 = a^2 + b^2, \text{ and } x = \pm\sqrt{(a^2+b^2)}.$$

164. Equations Reducible to Quadratics.—Sometimes equations of higher degrees may be *reduced to a quadratic form*, and solved as quadratics. Of this a few examples will now be given.

I. We have already seen that we can sometimes *discover a factor*, containing x which will give us *one* root, and will besides, enable us to reduce, by division, the equation to one of a *lower degree.*

Ex. 1. $x^3 + 3x^2 - 25x + 21 = 0$.

We see, on inspection, that this is satisfied when $x = 1$, $\therefore x - 1$ is an exact divisor, and gives the quotient

$$x^2 + 4x - 21 = 0,$$

which is a common quadratic.

Ex. 2. $x^3 - px = p - 1$.

This may be put in the form

$$x^3 + 1 = px + p,$$

or

$$x^3 + 1 = p(x + 1),$$

of which $x + 1$ is a divisor,

$$\therefore x + 1 = 0, \text{ or } x = -1;$$

and, by division, the equation reduces to

$$x^2 + x + 1 = p,$$

an ordinary quadratic.

Ex. 3. $x^4 - 3x^3 = 1 - 3x$; or, $x^4 - 1 = 3x^3 - 3x$.

Which gives us $(x^2 + 1)(x^2 - 1) = 3x(x^2 - 1)$,

where $x^2 - 1$ is a factor in each side,

$$\therefore x^2 - 1 = 0, \text{ or } x = \pm 1,$$

and the equation reduces to

$$x^2 + 1 = 3x,$$

a quadratic.

II. If the unknown quantity occurs in but two terms, and *its exponent in one of them is twice that in the other*, the equation may be solved as a quadratic.

Ex. 1. $x^4 - 9x^2 + 20 = 0$.

In this case we consider x^2 as the unknown quantity; put $x^2 = y$, then

$$y^2 - 9y + 20 = 0, \therefore y = 5 \text{ or } 4,$$

and $$x = \pm\sqrt{y} = \pm\sqrt{4} = \pm 2;$$

and $\pm\sqrt{5}$ are the other roots.

Ex. 2. $x^6 - 3x^3 - 28 = 0$.

Here we treat x^3 as the unknown quantity; let $x^3 = y$, then

$$y^2 - 3y - 28 = 0;$$

this gives us $$(y - 7)(y + 4) = 0,$$

$$\therefore \quad y = 7, \text{ or } -4;$$

but $$x^3 = y; \therefore x = \sqrt[3]{y} = \sqrt[3]{7}, \text{ or } \sqrt[3]{(-4)}.$$

Ex. 3. $x^n + \dfrac{6}{x^n} = 5$.

This gives $x^{2n} - 5x^n + 6 = 0$, in which we take x^n for the unknown quantity. Let it $= y$, so that

$$y^2 - 5y + 6 = 0$$

$$\therefore (y - 3)(y - 2) = 0,$$

and $$y = 3, \text{ or } 2,$$

but $$x^n = y; \therefore x = \sqrt[n]{y} = \sqrt[n]{3}, \text{ or } \sqrt[n]{2}.$$

Ex. 4. $(x^2 - 4)^2 - 8(x^2 - 4) + 15 = 0$.

Here we treat $x^2 - 4$ as the unknown quantity. Let it $= y$.

$$\therefore y^2 - 8y + 15 = 0$$

which gives, by factoring,

$$y = 3, \text{ or } 5,$$

$$\therefore x^2 - 4 = 3, \text{ or } = 5, \text{ etc.}$$

Ex. 5. $\left(\frac{x}{3}-\frac{4}{x}\right)^2=\frac{10}{3}\left(\frac{x}{3}-\frac{4}{x}\right)-\frac{8}{3}.$

Here we treat $\left(\frac{x}{3}-\frac{4}{x}\right)$ as the unknown quantity. Let it $= y$;

$$\therefore y^2-\frac{10}{3}y+\frac{8}{3}=0,$$

or
$$\left(y-\frac{4}{3}\right)(y-2)=0,$$

$$\therefore y=\frac{4}{3}, \text{ or } 2,$$

$$\therefore \frac{x}{3}-\frac{4}{x}=\frac{4}{3}, \text{ or } 2.$$

The first value, $\frac{4}{3}$, gives $x=6$, or -2;

the second value, 2, gives $x=3\pm\sqrt{(21)}$.

From these examples it appears that *any equation that can be put in the form* $ax^{2n}+bx^n+c=0$, *may be solved as a quadratic.*

Ex. 6. $x^4+x^3-2x^2-x+1=0.$

Divide through by x^2; then

$$x^2+x-2-\frac{1}{x}+\frac{1}{x^2}=0,$$

or
$$x^2-2+\frac{1}{x^2}+x-\frac{1}{x}=0,$$

or
$$\left(x-\frac{1}{x}\right)^2+\left(x-\frac{1}{x}\right)=0,$$

in which we treat $x-\frac{1}{x}$ as the unknown quantity.

$$\therefore \left(x-\frac{1}{x}\right)^2+\left(x-\frac{1}{x}\right)+\frac{1}{4}=\frac{1}{4},$$

or
$$x-\frac{1}{x}+\frac{1}{2}=\pm\frac{1}{2};$$

which gives two quadratics for determining x.

Ex. 7. $x^4 + ax^3 + bx^2 + ax + 1 = 0.$

Divide by x^2; $\quad \therefore x^2 + ax + b + \frac{a}{x} + \frac{1}{x^2} = 0,$

or $$x^2 + \frac{1}{x^2} + a\left(x + \frac{1}{x}\right) + b = 0.$$

Let $$x + \frac{1}{x} = y,$$

and $$\therefore x^2 + \frac{1}{x^2} + 2 = y^2,$$

or $$y^2 - 2 = x^2 + \frac{1}{x^2}:$$

hence the equation becomes

$$y^2 + ay + b - 2 = 0,$$

a common quadratic from which y may be found, and thence x.

EXERCISE LXXX.

(a)

Apply the general formula to obtain the values of x in the following equations:

1. $x^2 - 7x + 12 = 0$; $3x^2 - x - 2 = 0$; $4x^2 - 9x = 28$; $6x^2 + x - 35 = 0.$
4 or 3 ; 1 or $-\frac{2}{3}$; 4 or $-1\frac{3}{4}$; $2\frac{1}{3}$ or

2. $4x^2 - 9x = 28$; $x^2 + 6\frac{3}{5}x + 8 = 0$; $3x - \frac{14}{2x - 3} = 4x - 7.$
4 or $-1\frac{3}{4}$ $\quad$ -5 or $-1\frac{3}{5}$ $\quad$ 5 or $3\frac{1}{2}$

3. $2x + 1 - \frac{26}{2x + 1} = 11$; $35 + 9x - 2x^2 = 0$; $2 - 5x - 3x^2 = 0.$
6 or $-1\frac{1}{2}$ $\quad$ 7 or $-2\frac{1}{2}$ $\quad$ $\frac{1}{3}$ or -2

4. $26x - x^2 + 120 = 0$; × $\frac{x}{4} + \frac{25}{x} = 3$; $57x - 18x^2 + 145 = 0.$ ×
30 or -4 $\quad$ $6 \pm \sqrt{-64}$ $\quad$ $\frac{29}{6}$ or $-\frac{5}{3}$

5. $\frac{a}{x} + \frac{x}{b} = \frac{b}{x} + \frac{x}{a} + \frac{a^2 - b^2}{ab}$: $(a - b)(x^2 - b^2) = (a + b)(x - b)^2.$ ×
$x = a$ or b $\quad$ $x = a$ or b

6. $ax^2 - (c + d)x = bx^2 - \frac{cd}{a - b}$; $\frac{2x + 9}{x} + \frac{x}{2x + 9} = 5\frac{1}{5}.$ ×
$\frac{c}{a-b}$ or $\frac{d}{[illegible]}$ $\quad$ 3 or -5

7. $\frac{x+2}{x+3}+\frac{x-10}{x+8}=0$; $(a+b)(x-a)(x-b)=abx$.

2 or $-3\frac{1}{2}$ $a+b$ or $\frac{a}{a+b}$

(*b*)

Solve the following equations :—

1. $x^4+4x^2-12=0$; $4=5x^2-x^4$; $x^4+36=13x^2$.

$\pm\sqrt{2}, \pm\sqrt{-6}$ $\pm 1, \pm 2$; $\pm 2, \pm 3$

2. $x^4-x^2-156=0$; $x^4-14x^2+40=0$: $(7x)^2+9(7x)+18=0$.

$\pm\sqrt{13}, \sqrt{-12}$ $\pm 2 \pm\sqrt{10}$ $-\frac{3}{7}, -\frac{6}{7}$

3. $x^6+3x^3-88=0$; $(4x-1)^2-15(4x-1)+56=0$. $2, 2\frac{1}{4}$

$2, -\sqrt[3]{11}$

4. $x^6-35x^3+216=0$; $3x^4-7x^2=43076$. ± 11 $\pm\sqrt{-\frac{356}{3}}$

3, 2

5. $x^2-4x=8x+16$; $x^3-8=-(x^2-4)$; $x^3-x^2=-(x^3-1)$.

$-2, 4$ 2 $-3\pm\sqrt{-15}$ $1, \frac{-1\pm\sqrt{-7}}{4}$

6. $(x^2+4)^2-5(x^2+4)=44$; $(x^2+5)^2=7+x^2$.

$\pm\sqrt{\{+\frac{1}{2}(-3\pm\sqrt{201})\}}$ $x^2+5=2$ or -1 ∴ $x=\pm\sqrt{-3}$ or $\pm$

7. $256x^4-16x^2-2=0$; $x^6+6a^2x^3=16a^6$; $x^4-a^2x^2+b^4=0$.

$\pm\frac{\sqrt{-1}}{4}$, [illegible] ; $-2a$, ~~[illegible]~~ $a\sqrt[3]{2}$ $x^2=\frac{1}{2}\{a^2\pm\sqrt{(a^4-4b^4)}\}$

8. $x^6-7x^3=8$; $(4x)^4-20(4x)^2+99=0$; $x^{10}+31x^5=32$.

-1, ~~[illegible]~~, 2, ~~[illegible]~~ $\pm\frac{3}{4}$ $\pm\frac{1}{4}\sqrt{11}$ $1, -2$, [illegible]

9. $(x^2+7x+5)^2-x^2-7x-5=6$; $\left(\frac{12}{x}-x\right)^2+\frac{60}{x}-5x=6$.

$\frac{1}{2}\{-7\pm\sqrt{41}\}, \frac{1}{2}\{-7\pm\sqrt{21}\}$ $-4, 3, 3\pm\sqrt{21}$

10. $x^4+x^2+1=a(x^2+x+1)$; $x^3-7x^2+14x-8=0$.

$\frac{1}{2}\{1\pm\sqrt{4a-3}\}$; $\frac{-1\pm\ldots}{}$ 1, 2, 4

11. $4x^3+6x^2+x-1=0$; $2x^3+x^2-11x-10=0$; $2x^3-x^2=1$.

-1; $\frac{-1\pm\sqrt{5}}{4}$ $-1, -2, 2\frac{1}{2}$ $1, \frac{-1\pm\sqrt{-7}}{4}$

12. $x^3+x^2-12=0$; $x^3-x^2+3x-27=0$; $(x^2+1)(x+2)=2$.

$x=\frac{-3\pm\sqrt{-15}}{2}, x=2$ 3; $-1\pm\ldots$ $0, -1, -1$

13. $x^3-px^2+px-1=0$; $x^3-2px^2+2px=1$; $x^3-1=0$.

$1, \frac{1}{2}\{p-1\pm\sqrt{(p^2-2p-3)}\}$; $1, \frac{1}{2}\{2p-1\pm\sqrt{(4p^2-4p-3)}\}$; $x=1$

14. $x^3+x^2-4x-4=0$; $x^3+bx^2=a^2(a+b)$; $216x^7+19x^4=x$.

$-1, \pm 2$; a, [illegible] $0, \frac{1}{3}, -\frac{1}{2}$

15. $x^2+\frac{1}{x^2}+x+\frac{1}{x}=4$; $x^4+1\frac{1}{2}x^3-8x^2+1\frac{1}{2}x+1=0$.

$1, 1, \frac{1}{2}(-3\pm\sqrt{5})$ 2, [illegible] $-2\pm\sqrt{3}$

(*c*)

Solve the following equations :—

1. $1-\frac{x+5}{2x+1}=\frac{x-6}{x-2}$.

2. $\frac{2x+3}{2(2x-1)}-\frac{7-x}{2(x+1)}=\frac{7-3x}{4-3x}$.

3. $\frac{x-8}{x-5}+\frac{2(x+8)}{x+4}=\frac{3x+10}{x+1}$. $2, -22$

4. $\frac{x}{a}-\frac{a+b}{x}=\frac{a(4c-b)+4c^2}{a(a+2c)}$. $a+2c$ $-\frac{a(a+b)}{a+2c}$

5. $(x-3)^2+\frac{x-3}{x+3}=\frac{6}{(x+3)^2}$. $\pm\sqrt{6}, \pm\sqrt{11}$

6. $\frac{x}{x+1}+\frac{x}{x+2}+\frac{x}{x+5}+\frac{24}{3x+8}=3$. $0, -\frac{23}{13}$

7. $\frac{3x}{2(x+1)}-\frac{5}{8}=\frac{3x^2}{x^2-1}-\frac{23}{4(x-1)}$. $3, -1$

8. $\frac{12x^3-11x^2+10x-78}{8x^2-7x+6}=1\frac{1}{2}x-\frac{1}{2}$. $5, -4\frac{2}{7}$

9. $\frac{x+4}{x-4}+\frac{x+9}{x-9}=\frac{x-4}{x+4}+\frac{x-9}{x+9}$. $0, \pm 6$

10. $\frac{x^2+2x+2}{x+1}+\frac{x^2+8x+20}{x+4}=\frac{x^2+4x+6}{x+2}+\frac{x^2+6x+12}{x+3}$. $-2\frac{1}{2}$,

11. $\frac{ax+b}{bx+a}=\frac{cx-d}{dx-c}$. $x - \frac{ac-bd \pm\sqrt{(bd-ac)^2 - (ad-b}}{ad-bc}$,

12. $\frac{x^2+1}{4x^2-1}=\frac{x}{1+2x}-\frac{1}{4}$. $-\frac{3}{6}$

13. $\frac{1}{x^2+11x-8}+\frac{1}{x^2+2x-8}+\frac{1}{x^2-13x-8}=0$. ± 8 ± 1

14. $\frac{3x}{2}+\frac{81x^2-9}{(3x-1)(x+3)}=3x-\frac{3}{2}\cdot\frac{2x^2-1}{x+3}-\frac{57-3x}{2}$. $x=-2$

15. $\frac{7x^2+6x}{28}-\frac{2x^2+4\frac{2}{7}x}{23x-6}+\frac{x^2}{4}=\frac{11x^2}{21}-\frac{x^2-3x}{42}$. $4, 0$

16. $x^2+\frac{a-b}{ab^2}=\frac{14a^2-5b(a+2b)}{18a^2b^2}+\frac{2a-3b}{2ab}x$. $\frac{4a-5b}{6ab}$ $\frac{a-b}{3b}$

17. $(65x)^4+(65^2x)^2+1848^2=0$. $\pm\frac{3}{2}\sqrt{-1}$, $\pm\frac{6}{5}\sqrt{-1}$,

18. $(x^2-8x+11)^2+(x-4)^2=25$. $7, 1, 4$

19. $x^6 + 27 = 28x^3$. $1, 3$

20. $x^8 - 97x^4 + 1296 = 0$. $\pm 3\sqrt{\pm 1}, \pm 2\sqrt{\pm 1}$

21. $\dfrac{4x^3 + 4x^2 + 8x + 1}{2x^2 + 2x + 3} = \dfrac{2x^2 + 2x + 1}{x + 1}$. 2

22. $\dfrac{a}{x + a - c} + \dfrac{b}{x + b - c} = 2$. $c, c - \frac{1}{2}(a+b)$

23. $\dfrac{1}{5}\left(\dfrac{x - 4}{x - 9}\right) + \dfrac{1}{9}\left(\dfrac{x - 16}{x - 25}\right) - \dfrac{2}{13}\left(\dfrac{x - 36}{x - 49}\right) = \dfrac{92}{585}$. $x = 10$

24. $\dfrac{1}{(x + a)^2 - b^2} + \dfrac{1}{(x + b)^2 - a^2} = \dfrac{1}{x^2 - (a + b)^2} + \dfrac{1}{x^2 - (a - b)^2}$. $x = \dfrac{a^2+b^2}{a+b}$

25. $nx^3 + x + n + 1 = 0$. $-1, \frac{1}{2}\left(1 \pm \sqrt{\frac{-3n-4}{n}}\right)$

26. $x^4 + 2x^3 \overset{-}{+} x = 132$. $\frac{1}{2}\{-1 \pm \sqrt{46}\}$

27. $\dfrac{x^2 - a^2}{x^2 + x^2} + \dfrac{x^2 + a^2}{x^2 - a^2} = \dfrac{34}{15}$. $\pm 2a; \pm 2a\sqrt{-1}$

28. $\dfrac{3x^2 - 2x + 7}{6x^2 - 4x + 11} = \dfrac{x^2 + 7x - 3}{2x^2 + 14x - 9}$. $2, 2\frac{1}{2}$

29. $\dfrac{3x^2 + 10}{x} - \dfrac{x^2 + 2x + 3}{x + 2} = \dfrac{2x^2 + 2x + 10}{x + 1}$. $4, -\frac{5}{2}$

30. $\dfrac{2x + 1}{x + 2} - \dfrac{x + 2}{4x + 4} = \dfrac{7x + 8}{4x + 13}$. $8, -\frac{8}{7}$

(*d*)

Solve the following equations :—

1. $x^4 - 3\frac{1}{3}x^3 + 2x^2 - 3\frac{1}{3}x + 1 = 0$. $\pm\sqrt{-1}, 3, 1/3$

2. $x^3 \pm ax^2 \pm ax + 1 = 0$. (1) $-1, \dfrac{1 - a \pm \sqrt{a^2 - 2a - 3}}{2}$

3. $x^3 + 3\frac{1}{2}x^2 + 3\frac{1}{2}x + 1 = 0$. $-1, -2, -\frac{1}{2}$

4. $x^3 + 3x^2 - 6x - 8 = 0$. $2, -1, -4$

5. $\dfrac{x + 3}{x + 4} - \dfrac{x + 1}{x + 2} = \dfrac{4x + 9}{2x + 7} - \dfrac{12x + 17}{6x + 16}$. $-6, 4$

6. $\dfrac{4x+5}{x+1} + \dfrac{x+5}{x+4} = \dfrac{2x+5}{x+2} - \dfrac{x^2-10}{x+3} + x.$ $-2\frac{1}{2}$ ∞

7. $\dfrac{5x^2+x-3}{5x-4} - \dfrac{7x^2-3x-9}{7x-10} = \dfrac{x-3}{35x^2-78x+40}.$ $3, \infty$

8. $x^4 - 4\frac{1}{3}x^3 + 5\frac{1}{3}x^2 - 4\frac{1}{3}x + 1 = 0.$ $3, 1/3, \dfrac{1 \pm \sqrt{-3}}{2}$

9. $\dfrac{5}{x-1} + \dfrac{4}{x+2} + \dfrac{21}{x-3} = \dfrac{5}{x+1} + \dfrac{4}{x-2} + \dfrac{21}{x+3}.$ $x = \pm\sqrt{3}, \pm\sqrt{}$

10. $\dfrac{x^2-2x+3}{x^2+2x+4} + \dfrac{x^2+2x+4}{x^2-2x+3} = 2\frac{1}{2}.$

11. $\dfrac{x^4+1}{(1+x)^4} = 3\frac{1}{2}.$

12. $\dfrac{x+1}{x-1} + \dfrac{x-2}{x+2} + \dfrac{x-3}{x+3} + \dfrac{x+4}{x-4} = 4.$

13. $(5x^2+x+10)^2 - (4x^2+5x+8)^2 = (3x^2-x+5)^2 - (x^2+7x+1)^2.$

14. $x^3 + px^2 + (p-1)x + \dfrac{1}{p-1}(x+p-1) = 0.$

15. $(x^2-9x+18)(x^2-9x+20) = 24.$

16. $\dfrac{1+x+x^2}{1-x+x^2} = \dfrac{62}{63} \cdot \dfrac{1-x^3}{1+x^3}.$

17. $\left(\dfrac{a+x}{a-x}\right)^2 = 1 + \dfrac{cx}{b}.$

18. $\dfrac{1}{x-1} + \dfrac{1}{x-2} + \dfrac{1}{x-3} = \dfrac{3x^2}{(x-1)(x-2)(x-3)}.$

19. $\dfrac{x-1}{(x-2)(x-3)} + \dfrac{x-2}{(x-3)(x-1)} + \dfrac{x-3}{(x-1)(x-2)} = \dfrac{7}{6}.$

20. $\dfrac{40}{x^2+2x-48} - \dfrac{20}{x^2+9x+8} - \dfrac{8}{x^2+10x} + \dfrac{12}{x^2+5x+50} + 1 = 0.$

21. $\dfrac{x-4\frac{2}{5}}{3} - \dfrac{2x-3\frac{2}{3}}{4} = \dfrac{3}{2}\left(x - \dfrac{x-1\frac{1}{2}}{2}\right) + \dfrac{4x}{3}\left(x - 3 - \dfrac{(x-1)(x-2)}{x}\right).$

22. $\frac{1}{x}-\frac{1}{1+x}+\frac{1}{2+x}-\frac{1}{3+x}-\frac{1}{4+x}+\frac{1}{5+x}-\frac{1}{6+x}+\frac{1}{7+x}=0.$

23. $\frac{a}{x}+\frac{b}{x-1}+\frac{c}{x-2}+\frac{c}{x-3}+\frac{b}{x-4}+\frac{a}{x-5}=0.$

24. $\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}+\frac{1}{x+a+b-c}=0.$

25. $x^4+\left(n-\frac{1}{n}\right)x^3-2n^2x^2+\left(n-\frac{1}{n}\right)x+1=0.$

165. Solution of Problems.—We now give a few examples of *problems solved by quadratics.*

Ex. 1. A man bought a number of sheep and paid for each as many dollars as there were sheep; he sold $\frac{3}{5}$ of them at \$24 a head, and the rest at \$20 a head, making \$48 by the transaction. How many sheep did he buy?

Let $x =$ No. bought, then $\frac{3}{5}x$ at 24 + $\frac{2}{5}x$ at \$20 is \$48 more than *cost*, which is x^2. That is,

$$x^2=\frac{72x}{5}+\frac{40x}{5}-48.$$

Clearing and reducing

$$\therefore\quad 5x^2-112x+240=0;$$

or,

$$(x-20)(5x-12)=0;$$

$$\therefore\quad x=20.$$

Ex. 2. Two couriers start at the same time on a journey of 80 miles. A travels two miles an hour faster than B and arrives two hours before him. Find their rates of travelling.

Let $x =$ A's rate per hour, then $x-2 =$ B's rate.

Then $\frac{80}{x}=$ A's time and $\frac{80}{x-2}=$ B's time;

$$\therefore\quad \frac{80}{x-2}-\frac{80}{x}=2,$$

or $$80\left(\frac{1}{x-2}-\frac{1}{x}\right)=2, \quad \text{or} \quad 40\left(\frac{2}{x^2-2x}\right)=1;$$

$$\therefore \quad x^2-2x-80=0,$$

or $$(x-10)(x+8)=0;$$

whence $$x=10=\text{A's rate},$$

and $$10-2=\ 8=\text{B's rate}.$$

Ex. 3. A man bought a number of cows for \$540; if he had got 3 less for the same sum, the price of each would have been \$6 more. How many did he buy?

Let $x =$ the No., then $\frac{540}{x}$ = price of each, and $\frac{540}{x-3}$ = price of each had there been 3 less, and *this* is \$6 more than the actual price.

$$\therefore \quad \frac{540}{x}=\frac{540}{x-3}-6,$$

or $$540\left(\frac{1}{x-3}-\frac{1}{x}\right)=6,$$

or $$90\left(\frac{3}{x^2-3x}\right)=1,$$

whence we get $$(x-18)(x+15)=0,$$

and $$\therefore \quad x=18, \text{ or } -15;$$

of which only the former satisfies the conditions of the problem. The negative value may be considered to indicate a *diminution* of stock, and is a solution of the following problem: A man *sold* a number of cows for \$540; had he given 3 cows *more* for the same sum he would have received \$6 a head *less*. This problem is solved by the equation

$$\frac{540}{x}=\frac{540}{x+3}+6,$$

in which the values of x are 15 and -18.

EXERCISE LXXXI.

(*a*)

1. If to the square of a certain number 13 times itself be added, the sum will be 264. What is the number? 11 or -24

2. The area of a rectangle of which one side is 7 feet longer than the other is 494 feet. Find each side. 26 & 19

3. What number will give, when subtracted from its reciprocal, 6.09? $\frac{4}{25}$ or $-6\frac{1}{4}$

4. Divide a straight line *a* into two unequal parts so that one part shall be a mean proportional between *a* and the other part. $\frac{1}{2}(\sqrt{5}-1)a$, $\frac{1}{2}(3-\sqrt{5})a$

5. A garden plot 99 yards long and 66 broad is to be surrounded by a walk equal in area to the plot. Find the breadth of the walk. $16\frac{1}{2}$

6. A grocer bought $15 worth of coffee and paid an equal amount for raisins, receiving 10 lbs. more raisins than coffee; he sold 20 lbs. of coffee and 30 lbs. of raisins each at a profit of $33\frac{1}{3}$ per cent, for which he received in all $18. How many lbs. of each did he buy? 50 coffee, 60 raisins

7. By selling a horse for $432, a man gained a percentage equal to one-third of the number of dollars the horse cost him. Find what it cost him. $240

8. A plantation has 10,000 trees planted in rows; if the number of rows had been 20 less, there would have been 25 more trees in each row. How many rows were there? 100

9. A drover buys a number of cows for $540; if he had got 3 cows less for the same sum, the cost of each would have been $15 more. How many did he buy? 12

10. A tradesman bought a number of vases for $18; after breaking 4 of them he sold the rest at a profit of $2\frac{1}{4}$ each, and so neither gained nor lost. How many vases did he buy? 11

11. Two persons start together on a journey of 110 miles; A goes one mile an hour faster than B, and arrives one hour before him. Find their rates of travelling. A 11 B 10

12. The lengths in yards of the sides of a right-angled triangle are represented by three consecutive numbers. Find the length of each side. 3, 4, 5

13. What are oranges a dozen when two more for a dollar would lower the price a cent a dozen? 25¢

14. By lowering the price of apples a penny a dozen an apple-woman can sell 60 more for 5s. At what price did she sell them at first? 4d a dozen

15. Find two numbers whose difference is 15, and of which the cube of the less is equal to half their product. 3 & 18

16. A company at a tavern had \$35 to pay; two of them having left the room, the others had \$2 apiece more to pay. How many were in the company? 7

17. Two persons start from different places at the same time and travel towards each other; when they meet it is found that A has travelled 18 miles more than B, and that he could have done B's distance in $15\frac{3}{4}$ days, and that B would have been 28 days in doing A's distance. How far did each travel? A 72 B 54

18. An article is sold for \$9 at a loss of as much per cent as it is worth in dollars. Find its cost. \$90 or \$10

19. Find the quotient whose dividend is $2\frac{1}{2}$ less than the divisor, and which added to its reciprocal gives $2\frac{1}{2}$. $2\frac{1}{2}$

20. A and B put the sum of \$3,400 into business; A's money was in 12 months, and B's, 15 months; on settlement A received \$2,070 as capital and gain, and B received \$1,920. What capital did each invest? A [illegible]

21. Two pieces of cloth, consisting of 6 yards and 14 yards, respectively, are bought for \$53; and the buyer finds that for \$15 he gets one yard more of the latter than of the former. What is the price per yard of each piece? \$3

22. The diagonal of a rectangle, of which the breadth is *d* feet less than the length, measures *h* feet. Find the length and breadth of the rectangle. $\frac{1}{2}\{\sqrt{2h^2-d^2} \pm d\}$

23. Find the radius of a circle of which the area would be doubled by increasing its radius an inch. 2.414 inches

(*b*.)

1. A cistern has two pipes; to fill the cistern, one takes two hours less time than the other; both running together can fill it in $1\frac{7}{8}$ hours. How long will it take each to fill the cistern?

2. Two bodies move towards each other from two points 1800 yards apart, the first starting 5 seconds later than the second; but the first makes in each second 6 yards more than the second body, and they meet midway between the points. Find the velocity of each body per second.

3. A number has two digits, the left-hand digit being double of the right-hand digit; if the digits be reversed, the product of the number thus formed and the original number is 2268. Find the number.

4. A rectangular garden plot, of sides a and b, is surrounded by a path of uniform width and equal in area to one mth of the rectangle. Find the width of the path.

5. A company of men is formed into a hollow square 8 deep, and afterwards into a hollow square 4 deep, and it is found that the number of men in the front rank of the latter exceeds the number of men in the front rank of the former by 19. Find the number of men in the company.

6. A body of troops is drawn up in a column, the depth being 10 men more than the front of the column; but when drawn up in a line 5 deep, the front is 78 men more than it was. Find the number of men.

7. Some land was bought for \$1000; the buyer, keeping $1\frac{1}{2}$ acres for himself, sold the remainder at an advance of \$7 per acre, receiving for it \$1025. Find the number of acres bought, and the price per acre.

8. A and B start at the same instant from two points, P and Q, 60 yards apart; A, whose rate is 8 yards per second, runs at right angles to PQ; B, whose rate is 10 yards per second, runs in such a direction that, without changing it, he may just catch A. How long did they run?

9. If a carriage wheel $16\frac{1}{2}$ feet in circumference took one second more to revolve, the rate of the carriage per hour would be $1\frac{7}{8}$ miles less. At what rate is the carriage travelling?

10. There are two roads to a certain place; one, over the fields, is 8 miles, the main road is 11 miles. A man can ride a bicycle along the main road 10 miles an hour faster than he can walk over the other road, and would arrive 52 minutes sooner than by walking. Find his rate of walking.

11. A man climbs a mountain, walking $\frac{1}{2}$ a mile an hour faster during the first half of the distance than during the second, and reaches the top in $5\frac{1}{2}$ hours; he descends in $3\frac{3}{4}$ hours, walking a mile an hour faster than when he began the ascent. Find the distance to the top of the mountain.

12. An article is sold at a loss of as much per cent as it is worth in dollars. Show that it cannot be sold for more than $25.

13. A man begins business alone, but at the end of three months takes a partner, and their united capital is $1000; at the end of 2 months more the partnership is dissolved, and each gets $2000 as his share of the capital and profit. Find the original capital of each.

14. Out of each of two bags which contain a number of balls, a *handful* is taken, and the number remaining in the larger bag is equal to the cube of the number remaining in the smaller bag, and to the square of the number in the handful. A number is then taken out of the larger bag, such that the remainder is equal to the *square* of the number left in the smaller bag; the contents of the larger bag are now poured into the smaller bag, and it is found that the number in this bag is $\frac{5}{3}$ of the original number it contained. Find the number of balls in each bag at first, and the number that was taken out in the handful.

15. A person borrows $8200, and repays it, principal and interest, in two sums of $4410 at the end of the first and second years. Find the rate of interest.

16. A packet from Dover reaches Calais in 2 hours, but on the return voyage proceeds at first 6 miles an hour slower than

it went; but after half the return trip was made, the wind changed, and the packet sailed 2 miles an hour faster, and reached Dover sooner than it would have done, if the wind had not changed, in the proportion of 6 : 7. Find the distance between Dover and Calais.

17. A man can walk forwards 4 times as fast as he can backwards, and undertakes to walk a certain distance, $\frac{1}{4}$ of it backwards, in a certain time. But the ground being bad, his rate per hour backwards is $\frac{1}{2}$ mile less than he had expected, and he finds that to win his wager, he must walk forward 2 miles an hour faster. Find his rate per hour backwards.

18. A person sent money to his agent, to purchase sugar at $15 a barrel; the price having risen, he received 18 barrels less than he had expected; and it was found that 4 barrels more than $\frac{1}{3}$ of what he had expected would now cost $120 more than before. How many barrels were bought?

19. A person who copied a manuscript in a regular manner, found that the number of lines copied in the first half hour was less by 10 than the square root of the whole number of lines in the manuscript; and that the square of the number of lines copied in the first 49 minutes was equal to twice the number of lines then remaining to be copied. How many lines were there in the manuscript?

20. A and B engage to mow equal quantities of grass, and A began half an hour before B; they stopped at 12 o'clock, and rested an hour, when it was observed that just half the whole work was done. B's part was finished at 7 o'clock, and A's, at a quarter to 10. Supposing them to have worked uniformly, find the times at which they began.

CHAPTER XV.

SIMULTANEOUS QUADRATICS.

166. Two Unknowns.—We shall now give examples of the solution of *two equations, in one or both of which the unknown quantity is found in a higher degree then the first.* No general method can be given for the solution of such equations, but examples will be given of methods that are applicable in the more useful cases.

167. By Substitution.—*Find, from either of the equations, the value of one of the unknown quantities, in terms of the other, and substitute this value in the other equation.*

Ex. 1.
$$x^2 + \ y^2 = 25$$
$$4y + 3x = 24.$$

From the second equation $y = (24 - 3x) \div 4$. Substitute this in the first equation, and we have

$$x^2 + \left(\frac{24 - 3x}{4}\right)^2 = 25,$$

or
$$25x^2 - 144x + 176 = 0,$$

or
$$(25x - 44)(x - 4) = 0,$$
$$\therefore x = 4, \text{ or } \frac{44}{25}.$$

Substituting these values of x, successively, in the first equation, we get

$$4^2 + y^2 = 25, \text{ whence } y = 3\ ;$$

and
$$\left(\frac{44}{25}\right)^2 + y^2 = 25, \qquad \text{"} \qquad y = 4\tfrac{17}{25}.$$

Ex. 2.
$$x^2 + y^2 = 25$$
$$xy = 12.$$

From the second equation, $y = \dfrac{12}{x}$,

$$\therefore\ x^2 + \left(\frac{12}{x}\right)^2 = 25,$$

or
$$x^4 - 25x^2 + 144 = 0,$$

or
$$(x^2 - 9)(x^2 - 16) = 0,$$

$$\therefore\ x^2 = 16, \text{ or } 9; \text{ and } x = \pm 4, \text{ or } \pm 3.$$

And by substituting these values in the second equation, we find

$$y = \pm 3, \text{ or } \pm 4.$$

168. Homogeneous Equations.—When both equations are *homogeneous*, they may be solved *by treating the ratio* $\left(e.g., \frac{x}{y}\right)$ *as the unknown quantity. This method is most useful when both equations are of the same degree.*

Ex. 3. Solve the equations

$$x^2 + xy - 6y^2 = 6, \tag{1}$$

$$x^2 + 5xy + 6y^2 = 30. \tag{2}$$

Let $\frac{x}{y} = m$, that is $y = mx$, and substitute this value of y in each equation; thus

$$x^2 + mx^2 - 6m^2x^2 = 6, \tag{3}$$

$$x^2 + 5mx^2 + 6m^2x^2 = 30. \tag{4}$$

Dividing (4) by (3), and reducing, we have

$$36m^2 = 4; \qquad \therefore\ m = \pm \tfrac{1}{3}.$$

Substituting in (3) these values of m, we get

$$x = \pm 3; \ \therefore\ y = \pm 1.$$

Ex. 4.
$$x^3 + y^3 = 637, \tag{1}$$

$$x + y = 13. \tag{2}$$

y= m

Assume $x = my$, and substitute in the equations; thus

$$x^3 + m^3y^3 = 673, \quad \text{or} \quad x^3(1 + m^3) = 637, \tag{3}$$

$$x + my = 13, \quad \text{or} \quad x(1 + m) = 13, \tag{4}$$

Divide the *cube* of (4) by (3),

$$\therefore \quad \frac{(1+m)^3}{1+m^3} = \frac{13^3}{637} = \frac{169}{49};$$

$$\therefore \quad \frac{(1+m)^2}{1-m+m^2} = \frac{169}{49}.$$

From this equation m is readily found to be $\frac{5}{8}$ or $\frac{8}{5}$; and substituting the value of m in (4), we get $x = 5$ or 8, whence $y = 8$ or 5.

169. Elimination.—Where both the homogeneous equations are of the same degree, they are, in general, *most readily solved by eliminating the absolute terms*, as in finding the H. C. F. of two quantities.

Ex. 5. Take the equations in Ex. 3 (Art. 168),

$$x^2 + xy - 6y^2 = 6, \tag{1}$$

$$x^2 + 5xy + 6y^2 = 30. \tag{2}$$

(1) × 5 gives $\quad 5x^2 + 5xy - 30y^2 = 30, \quad (3)$

(3) − (2) gives $\quad 4x^2 - 36y^2 = 0.$

or $\quad x = \pm 3y,$

whence the values of x and y may be found.

Ex. 6. $\quad x^2 - 3xy - 10y^2 = 15,$

$$x^2 + 5xy + 6y^2 = 255.$$

Multiply the 1st equation by 17,

$$\therefore \quad 17x^2 - 51xy - 170y^2 = 255.$$

Subtract the 2d equation from this result,

$$\therefore \quad 16x^2 - 56xy - 176y^2 = 0,$$

or $\quad 2x^2 - 7xy - 22y^2 = 0;$

$$i.\,e., \quad (2x - 11y)(x + 2y) = 0;$$

$$\therefore \quad x = \tfrac{11}{2}y, \quad \text{or} \quad -2y, \text{ etc}$$

170. Symmetrical Equations.—When each of the equations is *symmetrical* in x and y, they may sometimes be *solved by such a substitution* as $u + v$ for x and $u - v$ for y.

Ex. 7.
$$x^3 + y^3 = 18xy, \qquad (1)$$
$$x + y = 12. \qquad (2)$$

Putting $u + v$ for x and $u - v$ for y, (1) becomes
$$(u + v)^3 + (u - v)^3 = 18\,(u + v)\,(u - v),$$
or
$$u^3 + 3uv^2 = 9\,(u^2 - v^2)\,; \qquad (3)$$
and (2) becomes $(u + v) + (u - v) = 12$,

which gives
$$u = 6.$$
Substituting this value of u in (3), we get
$$216 + 18v^2 = 9\,(36 - v^2),$$
or
$$27v^2 = 108\,;$$
$$\therefore\ v^2 = 4,$$
and
$$v = \pm 2\,;$$
hence
$$x = u + v = 6 \pm 2 = 8 \text{ or } 4,$$
and
$$y = u - v = 6 \mp 2 = 4 \text{ or } 8.$$

EXERCISE LXXXII.

(a.)

Solve the following equations by the method of substitution :—

1. $y = a + x$, $\pm b$
   $x^2 = b$. $a \pm b$

2. $x + y = a$, $\pm b$
   $x^2 - b^2 = 0$. $a \mp b$

3. $x^2 + y^2 = 10$, $x = 3, -\frac{13}{5}$
   $x - 2y = 1$. $y = 1, -1\frac{4}{5}$

4. $xy + y = 12$, $x = 3, -2\frac{1}{2}$
   $y - 2x = -3$ $y = 3, -8$

5. $x^2 + 5y = 41$, $x = 4, -10\frac{1}{4}$
   $5x - 4y = 0$. $y = 5, -12\frac{13}{16}$

×6. $x^2 - 4xy + 5y^2 = 13$, $x = \pm 7$
   $2x - 7y = 0$. $y = \pm 2$

× 7. $x + y = 9\,(x - y)$, $x = \pm 20$
   $xy = 320$. $y = \pm 16$

×8. $x - y = \frac{2}{3}\,(x + y)$, $x = \pm 15$
   $x^2 + xy = 270$. $y = \pm 3$

9. $x + y = 40$, x = 30, 10
$xy = 300$. y = 10, 30

10. $3x^2 - 2xy = 5$, x = 1, −5
$x - y = 2$. y = −1, −7

*11. $4y = 5x + 1$, x = 3, $-3\frac{1}{7}$
$2xy = 33 - x^2$. y = 4, $-3\frac{19}{28}$

×12. $x + \frac{1}{8} = \frac{2x + y}{3}$. x = 2, $-\frac{1}{3}$
$\frac{x + y}{x} = \frac{1}{2}(4x - y)$. y = 3, $\frac{2}{3}$

13. $x^2 + xy + y^2 = 7$, x = 1, $\frac{1}{7}$
$2x + 3y = 8$. y = 2, $2\frac{4}{7}$

14. $x^2 - y^2 = \frac{8}{17}(x^2 + y^2)$,
$5x + 7y = 92$. x = 10, 15
y = 6, −64

15. $x^2 + 4y^2 - 3x + y = 67$,
$x - 2y - 1 = 0$.
x = 7, $-4\frac{3}{4}$ y = 3, $-2\frac{7}{8}$

16. $21x^2 - 8xy = 53$,
$15x + 2y = 7$.
x = 1, $-\frac{53}{51}$ y = −4, $8\frac{11}{37}$

17. $5x^2 - 3y^2 + 118 = 0$, ± 7, ±
$99x^2 - 118xy + 35y^2 = 0$. ± 11, ±

18. $9x^2 + 5y^2 = 161$, x = 2, 3
$20x^2 + 6y^2 - 23xy = 0$. y = 5, 4

19. $x - \frac{1}{2}(x - y) = 4$, 2, 5,
$y - \frac{x + 3y}{x + 2} = 1$. 6, 3,

20. $x - y = 2$, 5, $\frac{3}{4}$,
$15(x^2 - y^2) = 16xy$. 3, $-1\frac{1}{4}$,

21. $3x^2 + 2y^2 = 165$, 7, 1 = x
$27x^2 + 7y^2 = 66xy$. 3, 9 = y

×22. $3x^2 + 4y^2 = 91$, x = ± √13
$2x^2 + 5y^2 = 7xy$. y = ± √1

23. $3x - 4y = 5$, x = 3, −15
$3x^2 - xy - 3y^2 = 21$. y = 1, $-\frac{38}{3}$

×24. $5x + y - 3 = 0$, x = 2,
$2x^2 - 3xy + y^2 = 1$. y = −7,

(*b*)

Solve the following homogeneous equations :—

1. $3x^2 + 11xy - 4y^2 = 126$,
$x^2 - 16y^2 = 9$.
x = ±5, y = ±1

2. $x^2 - 3xy - 10y^2 = 15$,
$x^2 + 5xy + 6y^2 = 255$.
x = ±11 ±2

3. $x^2 + 2xy + 3y^2 = 11$,
$x^2 + 6xy + 6y^2 = 22$.
x = 0, ±2 ±√(5/3), ±1

4. $14x^2 - 18xy + 5y^2 = 45$,
$x^2 - 3xy + y^2 = 9$.
x = 0, y = ±3 [illegible]

×5. $2y^2 - 4xy + 3x^2 = 17$, x = ±
$y^2 - x^2 = 16$. y = ± $\frac{1}{3}$

×6. $20x^2 + 7xy - 6y^2 = 21$, x = 0, ±1
$5x^2 + 4xy - 2y^2 = 7$. y = ±√−7

×7. $2x^2 - xy - 15y^2 = 7$, ± 2
$6x^2 + 11xy - 10y^2 = 35$, ± $\frac{1}{3}$

×8. $6x^2 - 5xy = 21$, x = ± ~~2~~ ±
$2x^2 + 3xy = 9$. y = ± ~~$\frac{3}{10}$~~

9. $6x^2 + xy + 4y^2 = 39,$
$2y^2 + 3xy = 9.$
$x = \pm 2\frac{1}{3}, \pm 1;\ y = \pm 1, \mp 3$

± 3 10. $4x^2 + 5y^2 = 216,$
± 6 $(2x - y)^2 - 144 = 0.$

11. $128x^2 + 48xy - 9y^2 = 143,$
$\mp 1\frac{1}{3}$ $272x^2 + 72xy - 9y^2 = 407.$

$\frac{a}{\sqrt{a+b}}$ 12. $x^2 + xy = a,$
$xy + y^2 = b.$ $y = \frac{\pm b}{\sqrt{a+b}}$

13. $x^2 - xy - 6y^2 = 24,$
2 $x^2 + 3xy - 10y^2 = 32.$

± 8 14. $x^2 + xy + 2y^2 = 74,$
5 $2x^2 + 2xy + y^2 = 73.$

15. $x^2 + xy + 4y^2 = 6,$ $\pm 2, \pm\sqrt{2}$
$3x^2 + 8y^2 = 14.$ $\pm\frac{1}{2}, \mp 2$

16. $x^2 - xy + y^2 = 21,$ $\pm 3\sqrt{3}$
$y^2 - 2xy + 15 = 0.$ $\pm 5, \pm\sqrt{3}$

17. $x^3 + y^3 = 407,$ 7, 4
$x + y = 11.$ 4, 7

18. $x^3 - y^3 = 218,$ 7, −5
$x - y = 2.$ 5, −7

19. $9x^2 + 9xy + y^2 = 319,$ $\pm 5, \pm$
$11(y^2 + 3x^2) + 3xy = 899.$ $\pm 2, 7$

20. $6xy - 2x^2 - y^2 = 31,$ $\pm 5, \pm 4$
$x^2 - 3xy + 3y^2 = 7.$ ± 3

Solve the following symmetrical equations :—

21. $(x + y)(x^2 + y^2) = 175,$
$x^2y + xy^2 = 84.$
$x = 4, 3;\ y = 3, 4$

9 22. $x + y = 23,$
$x^3 + y^3 = 3473.$

23. $x^3 + y^3 = 189,$ $x = 5, 4$
$x^2y + xy^2 = 180.$ $y = 4, 5$

24. $x^4 + y^4 = 272,$ 4, 2
$x + y = 6.$ 2, 4

25. $x^4 + y^4 = 337,$ 4
$x - y = 1.$ 3

26. $x^5 + y^5 = 275,$ 3, 2
$x + y = 5.$ 2, 3

27. $x^4 + y^4 = 22000,$ 13, 9
$x + y = 22.$ 9, 13

28. $x^4 + y^4 = 2145,$ 7, 4
$x - y = 3.$ 4, 7

29. $x + y = 4,$ 3, 1
$(x^2 + y^2)(x^3 + y^3) = 280.$ 1, 3

30. $x + y = 5,$ 3, 2
$(x^2 + y^2)(x^3 + y^3) = 455.$ 2, 3

171. Special Artifices.—The methods illustrated in the preceding articles may be applied to the classes of equations described; but many of these equations, and others which belong to none of these classes, may be solved by *special artifices*, of which a few examples will now be given.

Ex. 1. $$x + y = 12 \qquad (1)$$
$$xy = 35. \qquad (2)$$

(1) squared is $x^2 + 2xy + y^2 = 144,$ (3)

(2) × 4 is $4xy = 140,$ (4)

(3) − (4) gives $(x - y)^2 = 4,$

or $x - y = \pm 2.$ (5)

Now combine (5) and (1), and the values of x and y are easily found.

Ex. 2. $x^3 + y^3 = 189$ (1)

$\underline{x^2y + y^2x = 180.}$ (2)

(1) + 3 × (2) gives $x^3 + y^3 + 3x^2y + 3xy^2 = 729,$

or $(x + y)^3 = 729,$

$\therefore x + y = 9.$

Divide (2) by this, and we get $xy = 20.$

These equations can now be solved by substitution, or as in Ex. 1., giving $x = 5, 4$; $y = 4, 5$.

Ex. 3. $x^4y^3 - x^3y^4 = 216,$

or $x^3y^3(x - y) = 216,$ (1)

$x^2y - xy^2 = 6,$

or $\underline{xy(x - y) = 6,}$ (2)

(1) ÷ (2) gives $x^2y^2 = 36,$

or $xy = \pm 6$;

substitute in (2); and $y = 2$, or -3.

Ex. 4. $x - y = 4,$

$\underline{x^2 + y^2 = 106\,;}$

subtract the square of the *first* from the second; thus,

$$2xy = 90.$$

Add this to the second equation and we have

$$(x+y)^2 = 196,$$

or $$x + y = \pm 14;$$

combine this with the first equation, and x and y are easily found.

Ex. 5. $$x^4 + x^2y^2 + y^4 = 2188, \qquad (1)$$

$$\underline{x^2 + xy + y^2 = 76.} \qquad (2)$$

(1) ÷ (2) gives $$x^2 - xy + y^2 = 28, \qquad (3)$$

(2) − (3) " $$2xy = 48.$$

or $$xy = 24, \qquad (4)$$

(2) + (4) " $$(x+y)^2 = 100,$$

or $$x + y = \pm 10,$$

(3) − (4) " $$(x-y)^2 = 4,$$

or $$x - y = \pm 2, \text{ etc.}$$

Ex. 6. $$x - y = 3, \qquad (1)$$

$$\underline{x^5 - y^5 = 3093} \qquad (2)$$

(2) ÷ (1) gives $$x^4 + x^3y + x^2y^2 + xy^3 + y^4 = 1031,$$

or $$x^4 + y^4 + x^2y^2 + xy(x^2+y^2) = 1031, \qquad (3)$$

(1), squared, gives $$x^2 + y^2 - 2xy = 9,$$

or $$x^2 + y^2 = 9 + 2xy \qquad (4)$$

(3) is $$(x^2+y^2)^2 - x^2y^2 + xy(x^2+y^2) = 1031,$$

in this substitute for $x^2 + y^2$, from (4),

$$\therefore (9+2xy)^2 - x^2y^2 + xy(9+2xy) = 1031,$$

or $$x^2y^2 + 9xy = 190,$$

$$\therefore xy = 10, \text{ or } -19;$$

this, with (1), gives x and y.

Ex. 7
$$x^4 - x^2 + y^4 - y^2 = 84, \qquad (1)$$
$$x^2 + x^2y^2 + y^2 = 49, \qquad (2)$$

(1) is
$$x^4 + y^4 - (x^2 + y^2) = 84,$$
2 (2) is
$$2x^2y^2 + 2(x^2 + y^2) = 98, \qquad (3)$$
∴ adding (3) to (1), $(x^2 + y^2)^2 + (x^2 + y^2) = 182,$

a quadratic in terms of $x^2 + y^2$;

$$\therefore x^2 + y^2 = -\frac{1}{2} \pm \sqrt{\left(182 + \frac{1}{4}\right)} = -\frac{1}{2} \pm \frac{27}{2} = 13, \text{ or } -14;$$

these values being substituted in (2) give xy, etc.

Ex. 8
$$\frac{x+y}{x-y} + \frac{x-y}{x+y} = 7\tfrac{1}{7}, \qquad (1)$$
$$x^2y - xy^2 = 96; \qquad (2)$$

In (1) put
$$\frac{x+y}{x-y} = z,$$
and it becomes
$$z + \frac{1}{z} = 7\tfrac{1}{7},$$
or
$$z^2 - \tfrac{50}{7}z = -1,$$
which gives
$$z = 7 = \frac{x+y}{x-y}.$$
$$\therefore x = \tfrac{4}{3}y;$$

substitute, in (2), this value of x, and y is found to be 6.

EXERCISE LXXXIII.

(a)

Solve the following equations :—

1. $x^3 - y^3 = 9,$ $x = 1, 2$
$x - y = 3.$ $y = -2, -1$

2. $x - y = 7,$ $x = 4$ or 3
$x^2 + xy + y^2 = 13.$ $y = -3$ or -4

3. $x^2 - 2xy + 3y^2 = 1\tfrac{2}{9},$ $x = \pm \frac{7}{3\sqrt{19}}$
$x^2 + xy - y^2 = \frac{1}{9}.$ $y = \pm \frac{16}{3\sqrt{10}}$

4. $xy = 7,$ $x = \pm 1, \pm 7$
$x^2 + y^2 = 50.$ $y = \pm 7, \pm 1$

5. $x^2 + xy = 35$, $x = \pm 5$, [illegible]
$xy - y^2 = 6$. $y = \pm 2, \pm\frac{3}{\sqrt{2}}$

6. $x^2 + 4xy = 3$, $x = \pm 1$
$y^2 + 4xy = 2\frac{1}{4}$. $y = \pm\frac{1}{2}$

7. $x + y = a$, $x = \frac{a \pm 2b}{2}$
$4xy = a^2 - 4b^2$. $y = \frac{a \mp 2b}{2}$

8. $x^2 - y^2 - 4ab = 0$. $\pm(a + b)$
$xy - a^2 + b^2 = 0$. $\pm(a - b)$

9. $x^2 - xy = a^2 + b^2$, $\pm\frac{a^2 + b^2}{a - b}$
$xy - y^2 = 2ab$. $\pm\frac{2ab}{a - b}$

10. $\frac{1}{x} + \frac{1}{y} =$ $x = \frac{1}{2}$
$\frac{1}{x + 1} + \frac{1}{y + 1} = 1\frac{5}{12}$. $y = \frac{1}{3}$

11. $\frac{x + y}{x - y} + \frac{x - y}{x + y} = 2\frac{9}{40}$. $6\frac{1}{2}$
$6x - 9 = 20y$. $1\frac{1}{2}$

12. $x^3 - y^3 = a^3$, $0, a$
$x - y = a$. $a, 0$

13. $x^4 + x^2y^2 + y^4 = 9211$, $\pm 9, \pm 5$
$x^2 - xy + y^2 = 61$. $\pm 5, \pm 9$

14. $x^4 + x^2y^2 + y^4 = 2923$, $\pm 7, \pm 3$
$x^2 - xy + y^2 = 37$. $\pm 3, \pm 7$

15. $x^4 + x^2y^2 + y^4 = 481$, $\pm 4, \pm 3$
$x^2 + xy + y^2 = 37$. $\pm 3, \pm 4$

16. $x^5 - y^5 = 31$, $2, -1$
$x - y = 1$. $1, -2$

17. $x^4 - y^4 = 8080$, ✓
$x - y = 2$. ?

18. $x + y + xy = 11$, $-2, 3, 3 \pm \frac{1}{2}\sqrt{56}$
$xy^2 + x^2y = \mp 30$. $-3, 2, 3 \mp \frac{1}{2}\sqrt{}$

19. $xy + x - y = 9$, $5, -1, \frac{1}{2}(\pm\sqrt{41} + 5)$
$x^2y - xy^2 = 20$. $1, -5, \frac{1}{2}(\pm\sqrt{21} - 5)$

20. $x^2 + y^2 + 3(x + y) = 162$,
$xy = 12$.

21. $x^2 + y^2 + 3(x + y) = 44$, $x = \frac{1}{2}(9 \pm \sqrt{-}$
$x + y = 9$. $y = \frac{1}{2}(9 \mp \sqrt{}$

22. $x^2 + y^2 - 4x + 4y = 17$,
$x = 5, -2, -\frac{1}{2}(\,$[illegible]$\pm\sqrt{41})$
$xy = 10$.
$y = 2, -5$

23. $x^2 + 4y^2 + 6y - 3x = 292$,
$x = 17, -6, \frac{?}{} \pm \sqrt{118}$ [illegible]
$xy = 51$.
$y = 3, -8\frac{1}{2}$, [illegible] $\pm \frac{1}{2}\sqrt{118}$

24. $82xy = x^2y^2 + 1680$, $5, 6, 7, 8 = x$
$13 = x + y$. $8, 7, 6, 5$

25. $\frac{x + y}{x - y} = \frac{10}{3}$,
$x^2 + y^2 = 45$.

26. $4(x + y) = 3xy$, $4, 2$
$x + y + x^2 + y^2 = 26$. $2, 4$

27. $x + y = 11$, $1,$ [illegible]
$x^3 + y^3 = 1001$. [illegible]

28. $xy(x + y) = 30$, $3, 2$
$x^3 + y^3 = 35$. $2, 3$

29. $\frac{x^2}{y} + \frac{y^2}{x} = 18$, $8, 4$
$x + y = 12$. $4, 8$

30. $(3x + 4y)(7x - 2y) + 3x + 4y = 44$, $1, 1\frac{2}{17}$
$(3x + 4y)(7x - 2y) - 7x + 2y = 30$. $2, -\frac{1}{17}$

31. Eliminate x and y from the equations

$$ax^2 + bxy + cy^2 = 0,$$
$$a'x^2 + b'xy + c'y^2 = 0.$$

$(a'c - ac')^2 + (ab' - a'$[illegible]$)(bc' -$ [illegible]$) = 0$

172. More than two Unknowns.—We add a few examples of equations involving *more than two unknowns*, of which one at least is of the second degree, or of a higher degree than the second.

Ex. 1.
$$x^2 + y^2 + z^2 = 14, \quad (1)$$
$$x + y + z = 6, \quad (2)$$
$$xy + yz = 8, \quad (3)$$

(3) is
$$y(x + z) = 8;$$

in this, substitute the value of $x + z$ from (2),

$$\therefore y(6 - y) = 8,$$

or
$$y^2 - 6y + 8 = 0,$$

or
$$(y - 4)(y - 2) = 0,$$

$$\therefore y = 2, \text{ or } 4.$$

And (2) becomes, by putting for y the value 2,

$$x + z = 4; \quad (4)$$

$(2)^2 - (1)$ gives
$$2(xy + yz + xz) = 22, \quad (5)$$

$(5) - 2(3)$ gives
$$xz = 3;$$

which, combined with (4), can be solved by methods already given.

$$x = 1, \; y = 2, \; z = 3,$$

are values for x, y, z; and the other values may be found by substituting the second value of y.

Ex. 2.
$$x^2 + xy + xz = 36,$$
$$y^2 + xy + yz = 48,$$
$$z^2 + xz + yz = 60.$$

By adding the three equations, we get,

$$(x + y + z)^2 = 144,$$

or
$$x + y + z = \pm 12.$$

The first equation is $x(x + y + z) = 36$;

substitute in this the values of $x + y + z$,

$$\therefore x \times 12 = 36,$$

or $$x = 3;$$

the other value of $x + y + z$ (viz. -12) will give

$$x = -3;$$

similarly, from the other equations, y and z may be found.

$$x = \pm 3,\ y = \pm 4,\ z = \pm 5.$$

Ex. 3.
$$xy + yz + zx = a^2 - x^2, \tag{1}$$
$$xy + yz + zx = b^2 - y^2, \tag{2}$$
$$xy + yz + zx = c^2 - z^2, \tag{3}$$

(1) is $$x^2 + xy + yz + zx = a^2,$$

or $$x(x + z) + y(x + z) = a^2,$$

or $$(x + y)(x + z) = a^2. \tag{4}$$

And by symmetry,

(2) is $$(y + z)(y + x) = b^2, \tag{5}$$

(3) is $$(z + x)(z + y) = c^2, \tag{6}$$

$\therefore$ (4) × (5) × (6) gives

$$(x + y)^2 (y + z)^2 (z + x)^2 = a^2b^2c^2,$$

$$\therefore (x + y)(y + z)(z + x) = \pm abc, \tag{7}$$

(7) ÷ by (4), (5), (6), respectively, gives

$$y + z = \pm \frac{abc}{a^2},$$

$$z + x = \pm \frac{abc}{b^2},$$

$$x + y = \pm \frac{abc}{c^2};$$

Sol. [illegible] IV, 8

II − I

$y(4 - z - x) = 4$

$y = 4x + 4z$

[illegible]

$$\therefore \quad (x+y)+(z+x)-(y+z)=2x=\frac{abc}{c^2}+\frac{abc}{b^2}-\frac{abc}{a^2};$$

$$\therefore \quad x=\frac{abc}{2}\left(\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{a^2}\right);$$

and the values of y and z may be determined by symmetry.

EXERCISE LXXXIV.

Solve the following equations :—

1. $x+y+z=14$,
$x^2+y^2+z^2=84$,
$xz-y^2=0$.

2. $x^2+y^2+z^2=61$,
$xy+xz=36$,
$yz=18$.

3. $x+y+z=18$,
$x^2+y^2+z^2=110$,
$yz=30$.

4. $x^2+y^2+z^2=14$,
$x+y+z=6$,
$yz=6$.

5. $y+z=yz$,
$2(x+z)=3xz$,
$x+y=2xy$.

6. $2xy+3yz-xz=18$, $x=\pm\frac{4}{\sqrt{3}}$
$xy-2yz+3xz=16$, $y=\pm\sqrt{3}$
$3xy+yz-2xz=2$. $z=\pm 2\sqrt{3}$

7. $x+y+z=13$, $=7$ or 4
$x-y-1=0$, -8 or 3
← $xy \mp 2z=0$. 28 or 6

8. I $(x+z)(y-x-z)=3$,
II $(x+y+z)(y-x-z)=7$.
III $xy-z^2=0$.

$x=\pm 1$ $y=\pm$ [illegible] $z=\pm 2$

∓ 9 [illegible]

9. $xy+6(x+y)=36$, $x=2, -14$
$yz+6(y+z)=54$, $y=3, -15$
$zx+6(x+z)=44$. $z=4, -11$

10. $x^2+y^2+z^2=21$, $x=1, -2$
$xy+yz+zx=-6$, $y=-2, 1$
$x+y-z=-5$. $z=4$

11. $x^2+y^2+z^2=90$,
$x+y+z=6$,
$xz=28$.

12. $x+y+z=12$,
$xy+yz=27$,
$x^2+z^2=53$.

13. $xz+yz=1$,
$xy+xz=1$,
$xy+zy=1$.

14. $xy=a(x+y)$,
$xz=b(x+z)$,
$yz=c(y+z)$.

15. $x(x+y+z)=a^2$,
$y(x+y+z)=b^2$,
$z(x+y+z)=c^2$.

16. $x(y+z)=a$,
$y(z+x)=b$,
$z(x+y)=c$.

17. $2xy = c(x + y)$,
$2yz = a(y + z)$,
$2zx = b(z + x)$.

18. $xy + xz = 2a$,
$yz + yx = 2b$,
$zx + zy = 2c$.

19. $xy + n(x + y) = a^2 + 2an$,
$yz + n(y + z) = c^2 + 2cn$,
$zx + n(x + z) = b^2 + 2bn$.

20. $x(y + z - x) = a$,
$y(z + x - y) = b$,
$z(x + y - z) = c$.

21. $yz = a(x + y + z)$,
$zx = b(y + z + x)$,
$xy = c(x + y + z)$.

22. $x + y + z = 7$,
$x^2 + y^2 + z^2 = 21$,
$x^3 + y^3 + z^3 = 73$.

23. $(y + z)(x + y + z) = m$,
$(z + x)(x + y + z) = n$,
$(x + y)(x + y + z) = p$.

24. $(x + y)(z + x) = a$,
$(y + z)(x + y) = b$,
$(z + x)(y + z) = c$,

25. $x + y + (b - c)^2 xyz = 0$,
$y + z + (c - a)^2 xyz = 0$,
$z + x + (a - b)^2 xyz = 0$.

26. $y^2z^2 = a^3x$,
$x^2z^2 = b^3y$,
$x^2y^2 = c^3z$.

173. Problems.—We shall now give a few examples of *problems leading to simultaneous quadratics.*

Ex. 1. The numerator and denominator of one fraction are each greater by 1 than those of another, and the sum of the fractions is $1\frac{5}{12}$; if the numerators were interchanged, the sum of the fractions would be $1\frac{1}{2}$. Find the fractions.

Let $\frac{x}{y}$ denote one of the fractions; then $\frac{x+1}{y+1}$ will denote the other;

$$\therefore \quad \frac{x}{y} + \frac{x+1}{y+1} = 1\tfrac{5}{12}, \tag{1}$$

and
$$\frac{x+1}{y} + \frac{x}{y+1} = 1\tfrac{1}{2}; \tag{2}$$

(2) − (1) gives
$$\frac{1}{y} - \frac{1}{y+1} = \frac{1}{12},$$

or
$$y^2 + y = 12,$$

whence
$$y = 3,$$

and from (1),
$$x = 2;$$

hence $\frac{2}{3}$, $\frac{3}{4}$ are the fractions.

Ex. 2. A train travels 150 miles; had it travelled at the same rate for two hours, and the rest of the time at 40 miles an hour, it would have gone 210 miles. Find the time, and the rate per hour.

Let $x =$ time in hours, $y =$ rate per hour; then

$$xy = 150,$$

and $$2y + (x - 2)\,40 = 210.$$

From the second equation $y = 145 - 20x$;

substitute this in the first equation,

$$\therefore \quad x(145 - 20x) = 150,$$

or $$4x^2 - 29x + 30 = 0,$$

and $$x = 6;$$

whence $$y = 25.$$

Ex. 3. If $300 were allowed to remain at simple interest for a certain time it would amount to $360; if it were allowed to remain two years longer, at a rate of interest one per cent higher, it would amount to $405. Find the rate of interest and the time.

Let $x =$ No. of years, and $y =$ rate per cent; then

$$\frac{300 \cdot xy}{100} = 60,$$

or $$3xy = 60,$$

or $$xy = 20. \qquad (1)$$

Also $$\frac{300\,(x + 2)\,(y + 1)}{100} = 105,$$

or $$3\,(x + 2)\,(y + 1) = 105,$$

or $$(x + 2)\,(y + 1) = 35. \qquad (2)$$

By substituting the value of xy from (1),

(2) gives $$x = 13 - 2y,$$

and this value substituted in (1) gives

$$y\,(13 - 2y) = 20,$$

that is, $$2y^2 - 13y + 20 = 0.$$

or $$(2y - 5)(y - 4) = 0;$$

$$\therefore \quad y = 4, \text{ or } 2\tfrac{1}{2};$$

whence $$x = 5, \text{ or } 8.$$

Ex. 4. Find three numbers such that the sum of the first and second added to their product is 55, the sum of the second and third added to their product is 34, and the sum of the first and third added to their product is 39.

Let x, y, z denote the numbers; then

$$x + y + xy = 55, \tag{1}$$

$$y + z + zy = 34, \tag{2}$$

$$\underline{z + x + zx = 39.} \tag{3}$$

Add 1 to both sides of (1) and factor the result; then

$$(1 + x)(1 + y) = 56. \tag{4}$$

Similarly with (2) and (3),

$$(1 + y)(1 + z) = 35, \tag{5}$$

$$(1 + z)(1 + x) = 40; \tag{6}$$

$\therefore$ (4)·(5)·(6) gives

$$(1 + x)^2(1 + y)^2(1 + z)^2 = 40 \cdot 35 \cdot 56;$$

$$\therefore \quad (1 + x)(1 + y)(1 + z) = 280. \tag{7}$$

(7) ÷ (5) gives $$1 + x = 8;$$

$$\therefore \quad x = 7.$$

Similarly, $$1 + y = 7;$$

$$\therefore \quad y = 6.$$

And $$1 + z = 5;$$

$$\therefore \quad z = 5;$$

and the numbers are 7, 6, 4.

EXERCISE LXXXV.

PROBLEMS.

(*a*)

1. Find two numbers such that their sum, product, and difference of their squares shall be equal to one another.

2. Find two consecutive numbers such that the cube of the greater shall exceed the cube of the less by 183.

3. The product of two numbers is 576, and their quotient $2\frac{1}{4}$; find the numbers.

4. The product of two numbers is p, and their quotient q. Find the numbers.

5. Two smaller squares have the same area as a larger square whose side is 29 feet: one of the smaller squares contains 41 square feet more than the other. Find the sides of the smaller squares.

6. The product of the sum and difference of two numbers is a, the ratio of the sum of the numbers to their difference is $p:q$. Find the numbers.

7. The length, breadth, and depth of a rectangular stone are as $5:3:1$. The surface of the stone contains 2254 square inches. Find the dimensions of the stone.

8. Two cubes contain together 407 cubic inches; they measure together 11 inches in height. Find the contents of each cube.

9. If the numerator of a certain fraction be increased by 2, and the denominator be diminished by 2, the result will equal the reciprocal of the fraction. But if the numerator be diminished by 2 and the denominator be increased by 2, the result increased by $1\frac{1}{15}$ will equal the reciprocal of the required fraction. What is the fraction?

10. A certain number has two digits, whose product is equal to half the number: if the digits be inverted, and the new number be diminished by the original number, the

remainder will equal $1\frac{1}{2}$ times the product of the digits. Find the number.

11. Divide 102 into three parts such that the product of the first and third shall equal 102 times the second, and the third shall be $1\frac{1}{2}$ times the first.

12. A number consists of two digits whose product is 18; the difference between the square of the number and the number formed by inverting the digits is 2673. Find the number.

13. Find two numbers whose sum multiplied by the sum of their squares equals 488, and whose difference multiplied by the difference of their squares is 24.

14. The fore-wheel of a carriage turns, in a mile, 132 times more than the hind-wheel; if the circumference of each were 2 feet greater, it would turn only 88 times more. Find the circumference of each wheel.

15. Two boys run in opposite directions round a rectangular field of which the area is one acre; they start from one corner and meet 13 yards from the opposite corner, and the rate of one is $\frac{5}{6}$ that of the other. Find the dimensions of the field.

16. A ladder whose foot rests in a given position reaches, on one side of a street, a window 36 feet high, and when turned about its foot just reaches a window 27 feet high, on the other side of the street; if the two positions of the ladder be at right angles, find the width of the street, and the length of the ladder.

17. From two stations, A and B, 300 miles apart, two trains start at the same instant; the train from A reaches B 9 hours after they met, and the train from B reaches A 4 hours after they met. Find the rate of each train.

18. A person lends $1300 in two sums at different rates of interest, and receives equal returns from both; if the first portion had been lent at the second rate of interest it would have produced $36; and if the second portion had been lent at the first rate of interest it would have produced $49. Find the rates of interest.

19. The sides of a rectangular field are 119 yards and 19 yards, respectively; how much must the length be diminished and the breadth increased in order that the area may be the same, and the perimeter be 24 yards greater?

20. The diagonal of a rectangle is 20.4 feet; if the length of the rectangle be increased by 14 feet, and the breadth be diminished by 2.4 feet, the diagonal will be increased by 12.4 yards. Find the length and breadth.

21. A person bought for \$7500 a number of \$100 railway shares, when they were at a certain rate per cent discount; and afterwards, when they were at the same rate per cent premium, sold them, all but 60 of them, for \$5000; how many did he buy, and how much did he give for each share?

22. The difference of two numbers is 3, and the difference of their squares is 69. Find the numbers.

23. Two men sell a quantity of wheat for \$96; A sells 10 bushels more than B, and if he had sold the quantity B sold, would have received \$36 for it; while B would have received \$64 for what A sold. How much did each sell, and at what rate?

24. The sum of three numbers is 18; the sum of their squares is 134, and the product of the second and third is 50. Find the numbers.

25. The sum of three numbers is 12; the product of the first and second added to the product of the second and third is 32, and the sum of the squares of the first and third is 34. Find the numbers.

(*b*)

1. Three cubical blocks contain an aggregate of 73 cubic feet; the aggregate area of the faces is 126 sq. ft., and the sum of their depths is 7 feet. Find the dimensions of each block.

2. There is a number of three digits, the left-hand digit being double the right-hand digit; when the number is divided by the sum of its digits the quotient is 48; and when divided by the product of the two left-hand digits the quotient is 18. Find the number.

3. Find three numbers such that if the first be multiplied by the sum of the second and third, the second by the sum of the first and third, and the third by the sum of the first and second, the respective products shall be 26, 50, and 56.

4. A number consists of two digits; if it be multiplied by the number formed by inverting the digits, the product is 5092; if it be divided by the number thus formed, the quotient is 1, and the remainder a number of one digit. Find the number.

5. The *amount* of a certain capital at 8 per cent for a number of years was $2574; the *amount* of a capital $975 less, which was invested $12\frac{1}{2}$ years longer than the first capital and at the same rate, was $2574. Find the first capital, and the time it was invested.

6. There are three numbers such that the first multiplied by the sum of the other two is 68; the second multiplied by the sum of the other two is 98, and the third multiplied by the sum of the other two is 110. Find the numbers.

7. A, in running a race with B to a post and back, met him ten yards from the post; to make it a dead-heat, B must have increased his rate from this point $41\frac{3}{7}$ yards per minute; and if, without changing, he had turned back on meeting A, he would have come in four seconds after him. How far was it to the post?

8. The two sides of a rectangle expressed in feet have the sum of their cubes equal to 52 times their sum, and the difference of their cubes equal to 148 times their difference. Find the area and the diagonal.

9. There are three numbers; the sum of the first and second added to their product is 89; the sum of the first and third added to their product is 98; and the sum of the second and third added to their product is 109. Find the number.

10. Find a number greater than 100 and less than 1000 in which the middle digit is double of the right-hand digit and if the first and last digits be interchanged, the new number is 99 less than the original number; also the sum of the squares of the digits is 21.

11. The surface of a rectangular parallelopiped is 192 square feet; the length exceeds by 5 feet the sum of the breadth and height, and the diagonal is 13 feet. Find the length, breadth, and height.

12. Find three numbers such that 5 times the sum of the first and second added to their product is 63; 5 times the sum of the second and third added to their product is 129; and 5 times the sum of the first and third added to their product is 87.

13. Four numbers form a proportion; the sum of the means is a, the sum of the extremes b, and the sum of the squares of the four terms is c. Find the proportion.

14. There are three numbers; the product of the first and second added to their sum is a; the product of the first and third added to their sum is b; the product of the second and third added to their sum is c. Find the numbers.

15. A number of workmen can move a heap of stones in 8 hours from one place to another; if there had been 8 more workmen and each had carried 5 lbs. less at a time, the whole work would have occupied 7 hours. If there had been 8 fewer workmen and each had carried 11 lbs. more at a time, the work would have occupied 9 hours. Find the number of workmen and the weight each carried at a time.

16. In a proportion the sum of the means is a, the sum of the extremes is b, and the sum of the cubes of the four terms is c. Determine the proportion.

17. A rectangular field of which the length is 317 yards and breadth 119 yards, is divided into four parts by lines drawn parallel to its sides, so that the part lying in one corner of the field contains 8370 square yards, and that lying in the opposite corner, contains 10374 square yards. Find the length and breadth of each of these parts.

18. A three-figure number is equal to 42 times the sum of the middle and left hand digits, the sum of the digits is equal to 9, and the right hand digit is twice the sum of the other two. Find the number.

CHAPTER XVI.

THEORY OF QUADRATIC EQUATIONS.

174. *A quadratic equation cannot have more than two roots.*

For if possible let the equation

$$x^2 + px + q = 0$$

have three roots, α, β, γ. Then $x - \alpha$, $x - \beta$, and $x - \gamma$ are all factors of $x^2 + px + q$ (Art. 156), which is impossible, since $x^2 + px + q$ is of only the second degree in x.

Although a quadratic cannot have more than two roots, it may in effect have only one, by reason of the two roots being equal to one another. Thus, in

$$x^2 + px + q = 0,$$

if $q = \frac{p^2}{4}$, both roots are $-\frac{p}{2}$. In such cases analogy leads us to regard these not as one root but as two equal roots, since *two factors* of the form $(x - a)$ *are necessary to form the quadratic.*

175. We proceed to examine *certain forms which the roots of*

$$ax^2 + bx + c = 0$$

assume in consequence of relations between the coefficients of the equation, or of values of these coefficients :—

Solving the equation we have

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

(i) If $b^2 - 4ac$ be positive, the roots are *real* and *different.*

If $b^2 - 4ac = 0$, the roots are *real* and *equal.*

If $b^2 - 4ac$ be negative, the roots are *imaginary* and *different.*

(ii) If $c=0$, the roots are 0 and $-\frac{b}{a}$.

(iii) If $b=0$, the roots are $\pm\sqrt{-\frac{c}{a}}$, *i. e.*, equal but with opposite signs.

(iv) If $a=0$, the roots take the form $\frac{-b\pm b}{2a}$, *i. e.*, $\frac{0}{0}$ and ∞. The value of the expression $\frac{0}{0}$ in this case is determined by observing that the equation assumes the form

$$0\cdot x^2+bx+c=0,$$

which is satisfied by

$$x=-\frac{c}{b}.$$

Hence when $a=0$, the roots are $-\frac{c}{b}$ and infinity.

(v) If $a=0$ and $b=0$, the equation assumes the form

$$0\cdot x^2+0\cdot x+c=0.$$

Here, since the coefficients of x^2 and x are indefinitely small quantities, only an indefinitely large value for x could make the quantities $0\cdot x^2$ and $0\cdot x$ large enough to neutralize the c, and cause the whole expression ax^2+bx+c to vanish. Hence when $a=0$ and $b=0$, the roots are both infinite.

(vi) If $a=0$ and $c=0$, the roots assume the forms ∞ and $\frac{0}{0}$. The value of $\frac{0}{0}$ here is seen to be 0, since the equation takes the form

$$0x^2+bx+0=0.$$

Hence when $a=0$ and $c=0$, the roots are zero and infinity.

(vii) If $b=0$ and $c=0$, both roots become $\frac{0}{2a}$. Hence when $b=0$ and $c=0$, both roots are zero.

Examples of the disappearance of terms from equations are frequently met with, especially where the equation is presented in a fractional form. Thus in the equation

$$\frac{2x+5}{3x+2}-\frac{2x+4}{3x+7}=0,$$

on simplifying, the coefficient of x^2 is found to be zero, which, by iv, indicates that infinity is a root. And evidently $x=\infty$ satisfies the equation; for, putting it in the form

$$\frac{2+\frac{5}{x}}{3+\frac{2}{x}}-\frac{2+\frac{4}{x}}{3+\frac{7}{x}}=0,$$

and putting $x=\infty$, it becomes

$$\frac{2}{3}-\frac{2}{3}=0;$$

and the equation is satisfied.

176. We proceed to obtain certain *important relations between the roots of a quadratic and its coefficients.*

Solving $$ax^2+bx+c=0,$$

we get $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Let $$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},$$

and $$\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Then $$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2b}{2a}=-\frac{b}{a}.$$

Also $$\alpha\beta=\frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$=\frac{b^2-b^2+4ac}{4a^2}=\frac{c}{a}.$$

Hence, in a quadratic, when the coefficient of x^2 is made unity, the *coefficient of x with sign changed is equal to the sum* of the roots, and the *third term is equal to their product.*

We have
$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a\{x^2 - (\alpha + \beta)x + \alpha\beta\}$$
$$= a(x - \alpha)(x - \beta).$$

Hence on breaking a quadratic expression into factors, each factor equated to zero gives a root of the equation formed by equating the quadratic expression to zero,—a result previously obtained.

177. The preceding result also shows us how to form the equation whose roots are given. Thus, if 2 and -3 be the roots of an equation, the equation is

$$(x - 2)(x + 3) = 0,$$

or
$$x^2 + x - 6 = 0.$$

178. From the results

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a},$$

or (if the equation be $x^2 + px + q = 0$),

$$\alpha + \beta = -p, \quad \alpha\beta = q,$$

we may form, in terms of the coefficients, *expressions for symmetrical functions of the roots.*

Thus, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2q,$

$$\alpha^3 + \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2),$$
$$= (\alpha + \beta)\{(\alpha + \beta)^2 - 3\alpha\beta\} = -p(p^2 - 3q),$$
$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-p}{q}.$$

Ex. 1. Form the equation whose roots are the reciprocals of those of

$$x^2 + px + q = 0.$$

If α, β be the roots of this equation, the required equation will be

$$0 = \left(x - \frac{1}{\alpha}\right)\left(x - \frac{1}{\beta}\right),$$

$$= x^2 - \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)x + \frac{1}{\alpha\beta},$$

$$= x^2 - \frac{\alpha + \beta}{\alpha\beta}x + \frac{1}{\alpha\beta} = x^2 + \frac{p}{q}x + \frac{1}{q},$$

i. e., $qx^2 + px + 1 = 0.$

Ex. 2. Find the condition that one root of

$$ax^2 + bx + c = 0$$

may be twice the other.

Here $\alpha = 2\beta.$

Hence $-\frac{b}{a} = \alpha + \beta = 3\beta,$

and $\frac{c}{a} = \alpha\beta = 2\beta^2,$

$$\therefore \frac{c}{a} = 2\beta^2 = 2\left(-\frac{b}{3a}\right)^2;$$

and the required condition is

$$9ac - 2b^2 = 0.$$

Ex. 3. Find the condition that

$$x^2 + px + q = 0, \qquad (1)$$

and $x^2 + p'x + q' = 0 \qquad (2)$

may have a common root.

Let α be the common root.

Then $$\alpha^2 + p\alpha + q = 0, \tag{3}$$

$$\alpha^2 + p'\alpha + q' = 0; \tag{4}$$

(3) − (4) gives $$(p - p')\alpha + q - q' = 0. \tag{5}$$

(3) × q' gives $$q'\alpha^2 + pq'\alpha + qq' = 0, \tag{6}$$

(4) × q gives $$q\alpha^2 + p'q\alpha + qq' = 0. \tag{7}$$

(6) − (7) gives $$(q' - q)\alpha^2 + (pq' - p'q)\alpha = 0,$$

or $$(q' - q)\alpha + (pq' - p'q) = 0. \tag{8}$$

Equating the values of α obtained from (5) and (8) we have

$$(q - q')^2 + (p - p')(pq' - p'q) = 0,$$

the condition required.

Ex. 4. If α, β be the roots of

$$x^2 + px + q = 0,$$

and α, γ, the roots of $$x^2 + rx + s = 0,$$

then $\beta + \gamma$ is a root of the equation

$$x^2 + (p + r)x + 2(q + s) = 0.$$

Here $$\alpha + \beta = -p.\ \alpha\beta = q,$$

$$\alpha + \gamma = -r,\ \alpha\gamma = s,$$

$$\therefore 2\alpha + (\beta + \gamma) = -(p + r);$$

and $$\alpha(\beta + \gamma) = q + s.$$

Hence, $$2\frac{q + s}{\beta + \gamma} + (\beta + \gamma) + (p + r) = 0,$$

or $$(\beta + \gamma)^2 + (p + r)(\beta + \gamma) + 2(q + s) = 0,$$

and $\beta + \gamma$ is a root of

$$x^2 + (p + r)x + 2(q + s) = 0.$$

EXERCISE LXXXVI.

(*a*)

1. Form the quadratic whose roots are $1 \pm \sqrt{3}$. $x^2 - 2x - 2 = 0$

2. For what value of p will the equation $6x^2 - 5x + p = 0$ have equal roots? $p = \frac{25}{24}$

3. Form the equation whose roots are $-12\frac{3}{4}$ and $56\frac{7}{8}$.
$32x^2 - 1412x - 23205 = 0$

4. What quadratic has the roots $ab\sqrt{\left(\frac{a}{b}\right)}$ and $-ab\sqrt{\left(\frac{a}{b}\right)}$?
$x^2 - a^3b = 0$

5. If α, β are the roots of $x^2 - px + q = 0$, find the value of

   (1) $\alpha^2 + \alpha\beta + \beta^2$. (2) $\alpha^3 + \beta^3$. (3) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$. $\frac{p^2 - 2q}{q^2}$
   $p^2 - q$ $p(p^2 - 3q)$

6. Form the equation whose roots are $a - 2b$ and $3a - 4b$.
$x^2 - (4a - 6b)x + 3a^2 - 10ab + 8b^2 = 0$

7. The equation $x^2 - 4.2527x + 3.4906520649 = 0$ has one root 1.11111. Find the other root. 3.14159

8. For what values of x will $x^2 - 5x + 6\frac{1}{4}$ be
$x^2 - 5x + 6\frac{1}{4} = (x - \frac{5}{2})^2$ ∴ positive for all values of x except $\frac{5}{2}$
   (1) Positive? (2) Negative?

9. One root of the equation $x^2 + 444\frac{61}{66}x = 975406\frac{5}{66}$ is $-1234\frac{5}{6}$. Find the other root. $789\frac{10}{11}$

10. One root of the equation

$$x^2 - (5p - 7q + 9r)x + 18pr + 18r^2 = -4p^2 + 13pq - 10q^2 + 27qr$$

is $4p - 5q + 6r$. What is the other root? $p - 2q + 3r$

× 11. Show that in every quadratic of the form $ax^2 + bx + a = 0$, the roots are the reciprocals of each other.
$x^2 + \frac{b}{a}x + 1 = 0$ ∴ product of roots $= 1$

✓ 12. If α, β are the roots of $ax^2 + bx + c = 0$, and $\alpha^2 + \beta^2 = 1$, show that $b^2 - a^2 = 2ac$.

✓ 13. In the same equation, if $\frac{1}{3}(\alpha^2 + \beta^2) = \alpha\beta$, show that $b^2 = 5ac$.

✓ 14. If $x^2 - 3x = a$, and $x^2 - 4x = 5$, have a common root, prove that $a^2 - 14a + 40 = 0$.

✓ 15. Show that the roots of $cx^2 + bx + a = 0$ are the reciprocals of the roots of $ax^2 + bx + c = 0$.

✓ 16. If $x^2 = px + q$, show that

$$x^n = \frac{\alpha^n - \beta^n}{\alpha - \beta} x + q \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta},$$

where $\alpha + \beta = p$, $\alpha\beta = -q$.

(*b*)

✓ 1. If the roots of $ax^2 + bx + c = 0$ be equal, and those of $cx^2 + ax + b = 0$ be real, then b is intermediate in magnitude between a and $4c$.

2. Find the condition that the roots of $a^2x^2 + b^2x + c^2 = 0$ may be, respectively, the squares of the roots of $ax^2 + bx + c = 0$.

$b^2 - ac = 0$

✓ 3. If the quadratic $ax^2 + bx + c = 0$ be satisfied by more than two different values of x, it will be satisfied by all values of x.

4. If α, β be the roots of $x^2 + px + q = 0$, form the equations whose roots are

(1) $x^2 - (p^2 - 2q)x + q^2 = 0$

(2) $q^2x^2 - (p^2 - 2q)x + 1 = 0$

(1) α^2, β^2. (2) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$. (3) $\alpha^2 + \beta^2$, $\alpha\beta$. (4) $\alpha + \beta$, $\alpha - \beta$.

(3) $x^2 - (p^2 - q)x + q(p^2 - 2q) = 0$ (4) $x^2 + (p \mp \sqrt{p^2 - 4q})x \mp p$

✓ 5. If the equation $x^2 - ax + a + 15 = 0$ have equal roots, then $a = 10$ or -6.

6. If α, β be the roots of $x^2 + px + q = 0$, and $\alpha = \frac{1}{\beta^2}$, then $q^3 + pq + 1 = 0$.

✓ 7. If the roots of the equation $x^2 + px + q = 0$ be respectively equimultiples of the roots of the equation $x^2 + rx + s = 0$, then $sp^2 = qr^2$.

8. Find the conditions that the roots of $ax^2 + bx + c = 0$ may be the reciprocals of those of $a'x^2 + b'x + c' = 0$.

$a'b = b'c$ & $aa' = cc'$

✓ 9. In the equation $x^2 - a^2 + 4x + 3a + 2 = 0$, show that, if x have a real value, a cannot lie between 1 and 2.

✓ 10. If the difference between the roots of $ax^2 + 2bx + c = 0$ be equal to the difference between the roots of $a'x^2 + 2b'x + c' = 0$, then

$$\frac{b^2 - ac}{a^2} = \frac{b'^2 - a'c'}{a'^2}.$$

✓ 11. If $(a - b)x^2 + (a + b)^2 x + (a^2 - b^2)(a + b)$ have two equal roots, then $a = 3b$, or $3a = b$.

(*c*)

✓ 1. If $a^2x^2 + bx + bc + b^2 = 0$ have equal roots, then will

$$\frac{1}{4a^2} - \frac{c}{b} = 1.$$

✓ 2. Show that $(a - x)(b - x) - c^2 = 0$ will have equal roots if

$$(a - b)^2 + 4c^2 = 0.$$

✓ 3. If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio of m to n, show that

$$\frac{b^2}{ac} = \frac{(m + n)^2}{mn}.$$

4. In the equations $x^2 - 714x + 78165 = 0$ and $x^2 - 444x - 78165 = 0$, one root is the same in both, and the second root of the first equation, with its sign changed, is the second root of the second equation. Find the roots. (1) 579 & 135
(2) 579 & -135

5. If α, β are the roots of $x^2 + px + q = 0$, find the value of

(1) $\frac{1}{\alpha^3} + \frac{1}{\beta^3}$. (2) $\alpha^4 + \alpha^2\beta^2 + \beta^4$. (1) $\frac{-p(p^2-3q)}{q^3}$ (2) $(p^2-q)(p$

✓ 6. If α, β are the roots of $x^2 - (1 + a)x + \frac{1}{2}(1 + a + a^2) = 0$, show that $a = \alpha^2 + \beta^2$.

✓ 7. If α, β are the roots of $ax^2 + (a + b)x + b = 0$, show that

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{a}{b} + \frac{b}{a}.$$

✓ 8. If α, β are the roots of $x^2 - px + q = 0$, and α', β' are the roots of $x^2 - qx + p = 0$, show that

$$(\alpha + \beta)(\alpha' + \beta') \div \alpha\alpha'\beta\beta' = 1.$$

9. If α, β are the roots of $x^2 + ax + b = 0$, and α, γ, the roots of $x^2 + mx + n = 0$, then $\beta + \gamma$ is a root of the equation

$$x^2 + (a + m)x + 2(b + n) = 0.$$

10. If α, β are the roots of the equation $x^2 + ax + \frac{1}{4}(a^2 - b^2) = 0$, prove that $\alpha + \beta$, $\alpha - \beta$ are the roots of the equation

$$x^2 + (a \pm b)x \pm ab = 0.$$

11. If $x^2 + ax + b = 0$, and $x^2 + a'x + b' = 0$, have a common root, then

$$(ab' - a'b)(a - a') + (b - b')^2 = 0.$$

12. Find the condition that the roots of the equation

$$ax^2 + 2bx + c = 0$$

may be formed from the roots of

$$a'x^2 + 2b'x + c' = 0$$

by adding the same quantity to each root.

13. If the equations $ax + by = 1$, $cx^2 + dy^2 = 1$, have only one solution, prove that

$$\frac{a^2}{c} + \frac{b^2}{d} = 1, \quad \text{and} \quad x = \frac{a}{c}, \quad y = \frac{b}{d}.$$

179. Maximum and Minimum Values of an Expression.— The results of Art. 175 enable us to determine the *maximum* or *minimum* values of a quadratic expression.

Ex. 1. Find the minimum value of $x^2 - 3x + 5$.

Let $\qquad x^2 - 3x + 5 = m,$

or $\qquad x^2 - 3x = m - 5;$

$$\therefore \quad x = \tfrac{3}{2} \pm \sqrt{\tfrac{9}{4} + m - 5}$$

$$= \tfrac{3}{2} \pm \sqrt{(m - \tfrac{11}{4})};$$

from this result we see that for all *real* values of x, $m - \frac{11}{4}$ *must be positive; i. e.*, m cannot be *less* than $\frac{11}{4}$, which is therefore the minimum value of the given expression.

Ex. 2. Determine the maximum and minimum values of $4+x-x^2$.

Let $$4+x-x^2=m,$$

or $$x^2-x=4-m;$$

$$\therefore\ x=\tfrac{1}{2}\pm\sqrt{(4-m+\tfrac{1}{4})}$$
$$=\tfrac{1}{2}\pm\sqrt{(\tfrac{17}{4}-m)}.$$

In this result,

(1) m may be made *as small as we please*, and there is therefore no minimum value of the expression to which m is equal.

(2) But m *cannot be greater* than $\frac{17}{4}$, which is therefore the maximum value of the given expression.

Ex. 3. If x be real, show that $\dfrac{2x-7}{2x^2-2x-5}$ can have no real value between 1 and $\frac{1}{11}$.

Let $$\frac{2x-7}{2x^2-2x-5}=m:$$

$$\therefore\ 2mx^2-(2m+2)\,x-(5m-7)=0;$$

$$\therefore\ x=\frac{2m+2\pm\sqrt{\{(2m+2)^2+8m\,(5m-7)\}}}{4m}$$

$$=\frac{2m+2\pm2\sqrt{(11m^2-12m+1)}}{4m};$$

in which $11m^2-12m+1$, *i. e.*, $(11m-1)(m-1)$ must be positive if x is real; $\therefore$ $11m-1$ and $m-1$ must be positive, and $\therefore$ m cannot lie between $\frac{1}{11}$ and 1.

EXERCISE LXXXVII.

Find the maximum or minimum values of the following :—

1. x^2+4x+8. min 4
2. x^2-7x+8. " $-\frac{17}{4}$
3. $1+x-x^2$. max $\frac{5}{4}$
4. $x^2-\frac{2}{3}x+\frac{1}{6}$. min $\frac{1}{18}$
5. $50x^2+40x+9$. " 1
6. $72x^2-63x+64$. min $\frac{14463}{288}$ 1602
7. x^2+3x+4. $\frac{7}{4}$
8. $-x^2+4x-6$. -2 m
9. $x\div(1+x^2)$. max $\frac{1}{2}$ min
10. Divide a line 12 inches long into two parts so that the rectangle contained by them may be the greatest possible.

Bisect it.

16c

(b)

1. Show that $\dfrac{x^2 - 2x + 21}{6x - 14}$ can have no real values between 2 and $-\dfrac{10}{9}$.

2. What is the maximum area of any rectangle whose perimeter is 36 feet? 81 sq ft

3. A line a feet long is to be divided into two parts so that the sum of the squares on these parts may be the least possible. Bisect it

4. A right-angled triangle, with the hypothenuse a, is described so that its area is a maximum. Find the sides. Each side $\frac{1}{2} a \sqrt{2}$

5. Show that $(x - a)(b - x)$ can never exceed $\frac{1}{4}(a - b)^2$.

6. Determine the maximum value of $\dfrac{(x + a)(x - b)}{x^2}$. $\dfrac{(a+b)^2}{4ab}$

7. Determine the limits for the values of $\dfrac{n^2 - n + 1}{n^2 + n + 1}$. 3 & $\frac{1}{3}$

8. Find the greatest value which $\dfrac{a(x + a)}{x^2 + a^2}$ may have for any value of x. Omit $\dfrac{1}{\sqrt{2}} + \dfrac{1}{2}$

9. Show that the sum of the squares of the roots of the equation $x^2 + px + q = 0$ is the same as the sum of the squares of the roots of the equation $x^2 + 3px + q + 4p^2 = 0$.

10. If α, β are the roots of $ax^2 + bx + c = 0$, then will $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{b}{c} = 0$.

11. Find the condition that the difference of the roots of $ax^2 + bx + c = 0$, may be the same as the difference of the roots of $mx^2 + nx + r = 0$. $(b^2 - 4ac) \div a^2 = (n^2 - 4mr$

12. The equation $3px^2 + (6p - 12)x + 8$ has two equal roots; determine m. $p = 6$ or $\frac{2}{3}$

13. If α, β are the roots of $x^2 + px + q = 0$, and α', β' the roots of $x^2 - p'x + q' = 0$, then the equation whose roots are $\alpha'\beta + \beta'\alpha$ and $\alpha\beta + \alpha'\beta'$ is $x^2 - pp'x + p^2q' + p'^2q - 4qq' = 0$.

CHAPTER XVII.

INDEX LAWS.—SURDS.—SQUARES AND CUBES.

180. We shall now consider the general rules which govern the Multiplication, Division, Involution, and Evolution of *roots* and *powers*. These rules are given in the following **Index Laws.**

I. $a^m \times a^n = a^{m+n}$. III. $(ab)^n = a^n b^n$.

II. $(a^m)^n = (a^n)^m = a^{mn}$. IV. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$;

and the Law connecting the *Index* and the Surd Symbol is

$$\text{V.}\quad a^{\frac{m}{n}} = \sqrt[n]{a^m}.$$

181. Laws II, III, and IV are proved from Law I, which has already been proved.

1°. We have
$$(a^m)^n = a^m \cdot a^m \cdot a^m \ldots . \text{to } n \text{ factors}$$
$$= a^{m+m+m+\ldots . \text{to } n \text{ terms}} = a^{mn}.$$

Also
$$(a^n)^m = a^n \cdot a^n \cdot a^n \ldots . \text{to } m \text{ factors}$$
$$= a^{n+n+n+\ldots . \text{to } m \text{ terms}} = a^{mn}.$$

2°.
$$(ab)^n = ab \cdot ab \cdot ab \ldots . \text{to } n \text{ factors}$$
$$= a \cdot a \cdot a \ldots . \text{to } n \text{ factors} \times b \cdot b \cdot b \ldots \text{ to } n \text{ factors} = a^n b^n.$$

Hence, also, $(abc\ldots)^m = a^m b^m c^m \ldots .$

3°. $$\left(\frac{a}{b}\right)^n = \frac{a}{b} \cdot \frac{a}{b} \cdot \frac{a}{b} \ldots . \text{to } n \text{ factors} = \frac{a \cdot a \cdot a \ldots \text{to } n \text{ factors}}{b \cdot b \cdot b \ldots . \text{to } n \text{ factors}} = \frac{a^n}{b^n}.$$

182. In these proofs m and n are positive integers, and, by definition, denote the number of factors in each case, or the number of multiplications to be performed, beginning with unity as first multiplicand. We now assume these *index laws* to be true when m and n are *not* positive integers, but may be *negative integers*, or positive or negative *fractions;* and, assuming this, it becomes necessary to discover what *meanings* such indices will have. Consider the index as: 1°. *Fractional;* 2°. *Negative;* 3°. *Zero.*

1°. We have, by Law I,

$$a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a;$$

whence it appears that $a^{\frac{1}{2}}$ is one of the *two* equal factors which make a, *i. e.*, $a^{\frac{1}{2}}$ is a *square* (second) *root* of a. Similarly,

$$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a,$$

or $a^{\frac{1}{3}}$ is one of the *three* equal factors which make a, *i. e.*, $a^{\frac{1}{3}}$ is the *cube* (third) root of a: so, *generally*,

$$a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \cdot a^{\frac{1}{n}} \ldots \text{to } n \text{ factors} = a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n} \ldots \text{to } n \text{ terms}} = a^{\frac{n}{n}} = a,$$

or $a^{\frac{1}{n}}$ is one of the n equal factors which make a, *i. e.*, it is the nth *root* of a, or $a^{\frac{1}{n}} = \sqrt[n]{a}$. Similarly,

$$a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} \cdot a^{\frac{3}{4}} = a^{\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}} = a^{\frac{12}{4}} = a^3,$$

or $a^{\frac{3}{4}}$ is one of the *four* equal factors which make a^3, *i. e.*, it is the *fourth root* of a^3, or $a^{\frac{3}{4}} = \sqrt[4]{a^3}$.

So, *generally*,

$$a^{\frac{m}{n}} \cdot a^{\frac{m}{n}} \cdot a^{\frac{m}{n}} \ldots \text{to } n \text{ factors} = a^{\frac{m}{n}+\frac{m}{n}+\ldots \text{to } n \text{ terms}} = a^{\frac{mn}{n}} = a^m;$$

that is, $a^{\frac{m}{n}}$ is one of the n equal factors which make a^m, and is therefore the nth *root* of a^m, or

$$a^{\frac{m}{n}} = \sqrt[n]{a^m};$$

i. e., the numerator of a fractional index denotes a power, and the denominator, a root of the quantity affected by the index.

2°. By Law I, we have $a^m \cdot a^{-n} = a^{m-n}$.

But, also, $\frac{a^m}{a^n}$, *i. e.*, $a^m \cdot \frac{1}{a^n} = a^{m-n}$; $\therefore\ a^m \cdot \frac{1}{a^n} = a^m \cdot a^{-n}$;

and, dividing both sides by a^m, $\therefore\ \frac{1}{a^n} = a^{-n}$;

from which it appears that a quantity with *a negative index* *denotes the reciprocal of that quantity with the same positive index.* (See Art. 69.) Thus,

$$a^{-1}=\frac{1}{a},\quad a^{-2}=\frac{1}{a^2},\quad a^{-\frac{1}{3}}=\frac{1}{a^{\frac{1}{3}}}=\frac{1}{\sqrt[3]{a}},\text{ etc.}$$

It follows from this that a *factor* may be removed from the numerator to the denominator of a fraction, and conversely, by simply *changing the sign of its index.* Thus,

$$\frac{2a^3b}{3c^4d^2}=\tfrac{2}{3}a^3bc^{-4}d^{-2}=\frac{2}{3c^4d^2a^{-3}b^{-1}}.$$

3°. By Law I, we have $a^n\cdot a^0=a^{n+0}=a^n$;

but also $a^n\times 1=a^n;\ \therefore\ a^0=1,$

which corresponds with Arts. 30, 69.

Ex. 1. Express $\sqrt[3]{(a^5b^2)}+\sqrt[3]{(a^4b^5)}-\sqrt[4]{(a^5b^3)}$, with fractional indices.

The expression $=a^{\frac{5}{3}}b^{\frac{2}{3}}+a^{\frac{4}{3}}b^{\frac{5}{3}}-a^{\frac{5}{4}}b^{\frac{3}{4}}$.

Ex. 2. Remove the denominators from $\dfrac{a^3}{3b^2c^2}+\dfrac{4c^2}{a^2b}+\dfrac{2bc}{a}+\dfrac{1}{3abc}$.

The expression $=\frac{1}{3}a^3b^{-2}c^{-2}+4c^2a^{-2}b^{-1}+2bca^{-1}+\frac{1}{3}a^{-1}b^{-1}c^{-1}$.

Ex. 3. Express with surd symbols $\dfrac{a^{\frac{1}{4}}}{b^{\frac{3}{4}}}+\dfrac{a^{\frac{2}{3}}b^{\frac{1}{3}}}{2c^{\frac{1}{2}}}-\dfrac{2a^{\frac{1}{4}}c^{\frac{3}{4}}}{3b^{\frac{3}{2}}}$.

The expression $=\dfrac{\sqrt[4]{a}}{\sqrt[4]{b^3}}+\dfrac{\sqrt[3]{(a^2b)}}{2\sqrt{c}}-\dfrac{2\sqrt[4]{(ac^3)}}{3\sqrt{b^3}}$.

Ex. 4. Simplify $(3a-4b)^m\cdot(9a^2+16b^2)^m\cdot(3a+4b)^m$.

By Law III, the expression

$$=\{(3a-4b)\cdot(9a^2+16b^2)\cdot(3a+4b)\}^m=(81a^4-256b^4)^m.$$

Ex. 5. Simplify $\left(\dfrac{3a^2b^3c^4}{5d^4e^6f^7}\right)^m\div\left(\dfrac{9a^4b^2c}{25d^6e^7f^2}\right)^m$. By Law IV,

the expression $=\left(\dfrac{3a^2b^3c^4}{5d^4e^6f^7}\times\dfrac{25d^6e^7f^2}{9a^4b^2c}\right)^m=\left(\dfrac{5bc^3d^2e}{3a^2f^5}\right)^m$.

Ex. 6. $17^5 = 1,419,857$, find 34^5. $34 = 2 \times 17$;

$\therefore$, by Law III, $34^5 = 17^5 \times 2^5 = 17^5 \times 32 = 1,419,857 \times 32.$

Ex. 7. Simplify $\{(8x-6y)^{2a}\}^{5a} \div \{(4x-3y)^{5a}\}^{2a}$.

By Law II, we have the expression

$$= (8x-6y)^{10a} \div (4x-3y)^{10a} = \{(8x-6y) \div (4x-3y)\}^{10a} = 2^{10a}.$$

Ex. 8. Simplify $\sqrt[13]{[x^3y^2\sqrt[3]{\{xyz^4\sqrt[4]{(x^{-1}y^{-2}z^{-3})}\}}]^{12}}$.

Observe that the *cube* root of the *fourth* root, *i. e.*, the *twelfth* root, of $x^{-1}y^{-2}z^{-3}$, is to be taken; write with fractional exponents, and the expression

$$=\{x^3y^2\cdot x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{4}{3}}\cdot x^{-\frac{1}{12}}y^{-\frac{2}{12}}z^{-\frac{3}{12}}\}^{\frac{12}{13}} = (x^{\frac{13}{4}}\cdot y^{\frac{13}{6}}\cdot z^{\frac{13}{12}})^{\frac{12}{13}} = x^3y^2z.$$

EXERCISE LXXXVIII.

(*a*)

Simplify :—

1. $a^{\frac{1}{2}} \times a^{2a+b} \times a^{b-2a} \times a^{-\frac{1}{2}-2b}$.   2. $2a^0 \times 3b^0 \times 4c^0$. 24

3. $5(a^2)^0 \times (2b)^0 \times (3c^2)^0$.   4. $\frac{1}{a^0} \times \frac{2}{a^0} \times \frac{2a^0}{4}$.

Change the following to quantities under the root sign :—

5. $a^{\frac{2}{3}}$; $2a^{\frac{3}{4}}$; $5a^3b^{\frac{1}{2}}$; $7a^{\frac{1}{2}}b^{\frac{3}{2}}$; $6a^{\frac{1}{2}}b^{\frac{1}{4}}$; $a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$; $a^{\frac{3}{4}}b^{\frac{6}{7}}c^{\frac{10}{3}}$.

Change the following to quantities with fractional indices :—

6. $\sqrt{a}$; $\sqrt[3]{a^2b^4}$; $\sqrt[4]{a^2b^3c^{\frac{1}{2}}}$; $\sqrt[7]{a^2b^{-3}c}$; $\sqrt[5]{a^6b^2c^{-\frac{1}{3}}d^{\frac{1}{4}}}$.

Change the following so that (1) the denominator will be unity, (2) the numerator will be a whole number :—

7. $\frac{a^3}{b^3}$; $\frac{a^2bc}{a^3b^2}$; $\frac{a^{\frac{1}{2}}}{b^2}$; $\frac{7a^3b}{a^{\frac{1}{2}}b^{\frac{1}{4}}}$; $\frac{a^{-3}}{b^2}$; $\frac{a^2}{b^{-3}}$; $\frac{5a^2b^3}{a^{-3}b^{-4}c^5}$; $\frac{6a^{\frac{1}{2}}b^{\frac{1}{3}}}{a^{-\frac{3}{4}}b^{\frac{1}{6}}c^{\frac{7}{8}}}$.

(*b*)

1. Simplify : $(a-b)^n\cdot(a+b)^n$; $(x+y)^p \div (x+y)^q$; $(x^2-y^2)^n \div (x+y)^n$. a^2-b^2 $(x-y)^n$

2. Simplify : $3a^{\frac{1}{2}}\cdot a^{\frac{3}{2}}\cdot a^{\frac{4}{3}}\cdot 6a^2 \div 12a^{-\frac{5}{3}}\cdot a^{\frac{1}{2}}\cdot a^{\frac{5}{4}}$; $(x^{\frac{1}{2}}y^{-\frac{3}{2}})^{\frac{4}{3}}\cdot(xy^{-3})^{-\frac{2}{3}}$.

Express with fractional indices :—

$x^{\frac{4}{3}} - 4x^{\frac{1}{2}}$

3. $\sqrt[3]{x^4} - 3\sqrt[4]{x^2} - \sqrt{x}$; $\sqrt[3]{(a^2b^3)} - \sqrt[4]{(a^5b^8)} + 3\sqrt[5]{(a^{10}b^4)}$;
$a\sqrt[3]{b^8} + (\sqrt{a})^3 - \sqrt[3]{(a^6b)} - \sqrt[5]{(a^6b^4)}$.

4. $\sqrt[6]{(a^2b^8)} - 5\sqrt[5]{(a^5b^6)} + ab\sqrt[3]{(ab)} - a^2b^3\sqrt[5]{(a^4b)}$; $\sqrt[5]{a^{-8}} \div \sqrt[3]{a^8}$;
$3 \div \sqrt[3]{y^2}$; $\sqrt[4]{(x^3y^2)} \div \sqrt[5]{(x^{-7}y^{-10})}$.

$a^{\frac{1}{3}}b^{\frac{4}{3}} - 5ab^{\frac{6}{5}} + a^{\frac{4}{3}}b^{\frac{4}{3}} - a^{\frac{14}{5}}$

$a^{-\frac{4}{5}}$; $3y^{-\frac{2}{3}}$; $x^{\frac{43}{20}}y^{\frac{1}{2}}$

Express with positive indices :—

5. $a^{-2}b^{-3}c - ab^{-3} + a^{-4}bc^{-3} - a^{-1}b^{-2}c^{-3}$; $a^{-\frac{2}{3}}b^{-\frac{1}{3}}c + a^{-\frac{2}{3}}b^{\frac{2}{2}}c^{\frac{1}{3}} + b^{-\frac{3}{4}}$.

6. $\dfrac{a^{-2}b^{-3}}{c^{-3}} + \dfrac{2a}{b^{-1}c^{-1}} - \dfrac{3b^{-2}c^{-3}}{a^{-3}} + \dfrac{abc}{a^{-1}b^{-1}c^{-1}}$; $\dfrac{3a^{-3}x^2}{3y^2c^{-4}} - \dfrac{x^ay^{-b}}{x^{-b}y^a} + \dfrac{1}{5xy^{-4}}$.

Express with surd signs :

7. $x^{\frac{3}{5}} + y^{\frac{2}{3}} - z^{\frac{3}{5}}$; $x^{-\frac{2}{3}}\ y^{\frac{3}{4}}$; $7a^{-\frac{1}{2}} \cdot 3a^{-2\frac{1}{2}}$; $3a^{-\frac{1}{2}}b^{-\frac{2}{3}} \div 4a^{\frac{3}{4}}b^{-\frac{4}{3}}$.

8. $\dfrac{a^{-\frac{2}{3}}b^{-\frac{2}{3}}}{c^{-\frac{1}{3}}} + \dfrac{b^{-\frac{2}{3}}}{a^{-\frac{3}{2}}} - \dfrac{b^{-\frac{3}{4}}}{a^{-\frac{2}{3}}} + \dfrac{a^{-\frac{1}{2}}}{b^{-\frac{1}{2}}}$; $x^{\frac{1}{2n}} \cdot (a^2)^{\frac{3}{4}}$; $a^{\frac{3x}{4}} \div a^{\frac{2x}{3}}$; $a^{\frac{4}{n}} \cdot a^{\frac{3}{n}} \cdot a^{\frac{4}{n}}$.

9. Find the value of $32^{\frac{3}{5}}$; $4^{-\frac{5}{2}}$; $125^{-\frac{2}{3}}$; $8^{-\frac{4}{3}}$; $121^{\frac{1}{2}}$; $243^{-\frac{2}{5}}$.

10. $6^{10} = a$: find 12^{10} in terms of a; 18^5 in terms of a.

11. Simplify : $(2ab)^5 \cdot (3ab)^2 \cdot (5a)^4 \div \{(3b)^3 \cdot (4ab)^0\}$;
$(ab)^{x-2y} \cdot (ac)^{5x-6y} \cdot (bc)^{9x-10y}$.

12. Simplify : $(49x^2 - 36y^2)^{\frac{4}{3}} \div (7x + 6y)^{\frac{4}{3}}$; $(7x - 6y)^{\frac{4}{3}}$
$(5a^2 + 8ab - 21b^2)^x \div (a + 3b)^x$. $(5a - 7b)^x$

13. $5^6 = 15{,}625$, find 5^{12}; find the value of $(a^0)^n + (a^n)^0 + (a^0)^0 - b^0c^0$.

14. What value has $(x - y)^a \div a^{x-y}$ when $x = y$?

(*c*)

Simplify the following :—

1. $(a^{-\frac{2}{3}}b^{\frac{1}{6}})^{\frac{4}{3}}$; $\{(a^{-\frac{3}{4}} \cdot b^{-\frac{4}{3}})^{\frac{3}{4}}\}^{-\frac{4}{3}}$; $\left(\dfrac{27x^{-6}}{8a^{-6}}\right)^{-\frac{4}{3}}$; $\sqrt[4]{(x^3\sqrt[3]{x^{-4}})}$.

$b^{\frac{2}{9}} \div a^{\frac{8}{9}}$; $a^{\frac{3}{4}}b^{\frac{4}{3}}$; $81x^8 \div$ [illegible] a^8; $x^{\frac{5}{12}}$

2. $(x \div \sqrt[2n]{x})^{2n}$; $\{\sqrt[4]{(a^{-4}b^2)} \cdot \sqrt[3]{(a^{-2}b^{-3})}\}^{-\frac{3}{4}}$; $\sqrt[3]{\{(x-y)^{-3} \cdot (x-y)^{-3}\}}$.

x^{2n-1} $a^{\frac{5}{4}}b^{\frac{3}{8}}$ $\dfrac{1}{(x-y)^2}$

3. $\left(\frac{a^{-4}b^{2}}{a^{4}b^{-4}}\right)^{-3} \div \left(\frac{a^{-1}b^{-1}}{a^{-3}b^{-4}}\right)^{5}$; $\{a^{-\frac{1}{3}}b^{-\frac{1}{3}}\sqrt{(ab^{-\frac{1}{3}}\sqrt[4]{b^{-\frac{4}{3}}})}\}^{\frac{2}{3}}$; $a^{11}b^{-33}$;
$\left(\frac{x^{-2}y^{3}}{x^{3}y^{-2}}\right)^{-\frac{1}{5}} \times \left(\frac{y^{3}x^{-3}}{x^{3}y^{-3}}\right)^{-1}$. $a^{\frac{1}{9}}b^{-\frac{4}{9}}$; $\frac{x}{y}{}^{7}$

4. $(a^{m}b^{n})^{\frac{1}{m}} \times (a^{n}b^{m})^{\frac{1}{n}}$; $(a^{-m}b^{n})^{-\frac{1}{m}} \times (a^{n}b^{-m})^{-\frac{1}{n}}$; $a^{2}b^{\frac{m^{2}+n^{2}}{mn}}$; $b^{\frac{m^{2}-}{m}}$
$\{(a^{m})^{-n}\}^{p} \times \{(a^{-2m})^{3n}\}^{-2p}$; $\{(ab)^{-m}\}^{n} \div (a^{n}b^{m})^{-m}$. a^{11mnp}; b

5. $\sqrt{\{a^{-1}\sqrt{(a^{3}\sqrt{a^{-4}})}\}}$; $[\sqrt{\{a^{2}b^{3}\sqrt{(a^{4}b^{-2})}\}}]^{6}$; $a^{-\frac{1}{4}}$; $a^{12}b^{6}$; x^{-4}
$\{x^{-\frac{8}{2}}y(xy^{-2})^{-\frac{1}{2}}(x^{-1}y)^{-\frac{2}{3}}\}^{3}$; $\{(ab)^{m}\}^{\frac{1}{n}} \div (a^{-n}b^{-m})^{\frac{1}{m}}$. $a^{\frac{m^{2}+n^{2}}{mn}}$ b

6. $\sqrt[13]{[a^{6}b^{4}\sqrt[3]{\{a^{2}b^{2}c^{8}\sqrt[4]{(a^{-2}b^{-4}c^{-8})}\}}]^{12}}$; $a^{6}b^{4}c^{2}$; $a^{2}b^{6}c$
$\sqrt[m]{(a^{2m-n}b^{5m+1}c^{3p})} \cdot \sqrt[m]{(a^{n}b^{m-1}c^{m-3p})}$.

7. $\sqrt[10]{\{-a^{5}\cdot(-b)^{4}\cdot(-c)^{-3}\}^{5}}$; $\{x^{2}\cdot(-y^{6})^{\frac{1}{3}} \div -y^{3}(-x^{2})^{\frac{1}{2}}\}^{6}$; $\frac{a^{\frac{5}{2}}}{c^{\frac{3}{2}}}$
$\{ab^{2}\cdot(ab^{3})^{\frac{1}{2}}\cdot(ab^{4})^{\frac{1}{3}}\cdot(ab^{5})^{\frac{1}{4}}\cdot a^{39}b^{-3}\}^{\frac{1}{3}}$. $-\frac{x^{6}}{y^{6}}$; $a^{\frac{493}{36}}b^{\frac{37}{36}}$

183. Since the Index Laws are universally true, the rules which govern the ordinary operations of Multiplication, Division, Involution, Evolution, Resolution into factors, are applicable *whatever* be the indices. A few examples of this will now be given.

Ex. 1. Multiply $-a^{\frac{1}{5}} + 3a^{\frac{3}{5}} - 4a^{\frac{1}{5}}$ by $3a^{\frac{1}{5}} - 6a^{-\frac{3}{5}} + a^{-\frac{1}{5}}$.

Arrange both factors in descending powers of a :—

$$\begin{array}{rrrrr}
3a^{\frac{3}{5}} - & 4a^{\frac{1}{5}} - & a^{-\frac{1}{5}} & & \\
3a^{\frac{1}{5}} + & a^{-\frac{1}{5}} - & 6a^{-\frac{3}{5}} & & \\
\hline
9a^{\frac{4}{5}} - & 12a^{\frac{2}{5}} - & 3a^{0} & & \\
+ & 3a^{\frac{2}{5}} - & 4a^{0} & - a^{-\frac{2}{5}} & \\
 & & - 18a^{0} & + 24a^{-\frac{2}{5}} & + 6a^{-\frac{4}{5}} \\
\hline
9a^{\frac{4}{5}} - & 9a^{\frac{2}{5}} - & 25 & + 23a^{-\frac{2}{5}} & + 6a^{-\frac{4}{5}}.
\end{array}$$

Ex. 2. $(x^{\frac{3}{2}} - 1 + x^{-\frac{3}{2}})(x^{\frac{3}{2}} + 1 + x^{-\frac{3}{2}})$.

Here formula (B) applies ; the expression

$$= \{(x^{\frac{3}{2}} + x^{-\frac{3}{2}}) - 1\}\{(x^{\frac{3}{2}} + x^{-\frac{3}{2}}) + 1\} = (x^{\frac{3}{2}} + x^{-\frac{3}{2}})^{2} - 1$$
$$= x^{3} + 2x^{0} + x^{-3} - 1 = x^{3} + 1 + x^{-3}.$$

Ex. 3. $\{(ab)^{\frac12}+(ac)^{\frac12}+(bd)^{\frac12}+(cd)^{\frac12}\}\{(ab)^{\frac12}-(ac)^{\frac12}-(bd)^{\frac12}+(cd)^{\frac12}\}$.

Here we have $\{(ab)^{\frac12}+(cd)^{\frac12}\} + \{(ac)^{\frac12}+(bd)^{\frac12}\}$

to be multiplied by $\{(ab)^{\frac12}+(cd)^{\frac12}\} - \{(ac)^{\frac12}+(bd)^{\frac12}\}$.

and the product $= \{(ab)^{\frac12}+(cd)^{\frac12}\}^2 - \{(ac)^{\frac12}+(bd)^{\frac12}\}^2$

$$= ab + cd + 2(abcd)^{\frac12} - ac - bd - 2(abcd)^{\frac12}$$

$$= ab - ac - bd + cd = a(b-c) - d(b-c)$$

$$= (a-d)(b-c).$$

Ex. 4. $(x^{\frac43} - y^{\frac43}) \div (x^{\frac13} + y^{\frac13}) = x^{\frac33} - x^{\frac23}y^{\frac13} + x^{\frac13}y^{\frac23} - y^{\frac33}$.

$$: x - x^{\frac23}y^{\frac13} + x^{\frac13}y^{\frac23} - y.$$

Ex. 5. $(16x - y^2) \div (2x^{\frac14} - y^{\frac12})$.

This is $\{(2x^{\frac14})^4 - (y^{\frac12})^4\} \div (2x^{\frac14} - y^{\frac12})$

$$= 8x^{\frac34} + 4x^{\frac12}y^{\frac12} + 2x^{\frac14}y + y^{\frac32}.$$

Ex. 6. Resolve $x^2 - 6x - 2$.

Complete the square by adding $+ 9 - 9$, and we have

$$x^2 - 6x + 9 - 2 - 9 = (x-3)^2 - 11$$

$$= \{(x-3) + \sqrt{11}\}\{(x-3) - \sqrt{11}\}.$$

Ex. 7. Resolve $x^{\frac45} + x^{\frac25}y^{\frac25} + y^{\frac45}$.

Add the complementary terms $x^{\frac25}y^{\frac25} - x^{\frac25}y^{\frac25}$, and we have

$$(x^{\frac25} + y^{\frac25})^2 - x^{\frac25}y^{\frac25} = (x^{\frac25} + y^{\frac25} + x^{\frac15}y^{\frac15})(x^{\frac25} + y^{\frac25} - x^{\frac15}y^{\frac15}).$$

Ex. 8. Resolve $3ax^3 - 2a^{\frac23}x^2 - a^{\frac13}x$.

The expression $= a^{\frac13}x(3a^{\frac23}x^2 - 2a^{\frac13}x - 1)$,

which, using formula (D), $= a^{\frac13}x(a^{\frac13}x - 1)(3a^{\frac13}x + 1)$.

Ex. 9. Reduce $\dfrac{x^2 - x^2y^{-2} - 1 + y^{-2}}{x + xy^{-1} + 1 + y^{-1}}$. This expression

$$= \frac{x^2(1-y^{-2}) - (1-y^{-2})}{x(1+y^{-1}) + (1+y^{-1})} = \frac{(1-y^{-2})(x^2-1)}{(1+y^{-1})(x+1)} = (1-y^{-1})(x-1)$$

Ex. 10. Reduce $\dfrac{1 + x^{\frac{1}{2}} - x^{\frac{3}{2}} - x^2}{x^{\frac{3}{2}} + 6x^2 + 5x^{\frac{5}{2}}}$. The expression

$$= \frac{1 - x^{\frac{3}{2}} + x^{\frac{1}{2}}(1 - x^{\frac{3}{2}})}{x^{\frac{3}{2}}(1 + x^{\frac{1}{2}}) + 5x^2(1 + x^{\frac{1}{2}})} = \frac{(1 - x^{\frac{3}{2}})(1 + x^{\frac{1}{2}})}{(x^{\frac{3}{2}} + 5x^2)(1 + x^{\frac{1}{2}})} = \frac{1 - x^{\frac{3}{2}}}{x^{\frac{3}{2}} + 5x^2}.$$

EXERCISE LXXXIX.

Simplify :— (11) x^{a+b+c}; x^{a^2+a+1}; $x^{m^2n^2}$; $a^{\frac{m^3}{m+1}}$

1. $x^{a+b-c} \cdot x^{b+c-a} \cdot x^{c+a-b}$; $x^{a^2} \cdot x^a \cdot x$; $\{(x^m)^{-n}\}^{-mn}$; $\{(a^m)^m\}^{\frac{m}{m+1}}$.

2. $\dfrac{2^n \times (2^{n-1})^n}{2^{n+1} \times \cdot 2^{n-1} \times \cdot 4^{-n}}$; $\dfrac{3^{n+2}}{(3^{n+1})^n} \div \dfrac{9^{n+2}}{(3^n)^{n+2}}$; $\left\{\left(\dfrac{a^{-m}}{b^{-n}}\right)^{\frac{p}{m}}\right\}^{-\frac{q}{n}}$.

$2^{n^2} \div 10$; $\frac{1}{9}$; $(a^{\frac{1}{n}} b^{-\frac{1}{m}})^{pq}$

Multiply :—

3. $(x^3 + x^2 + 1)(x^{\frac{3}{2}} + x + 1)$; $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{2}{3}} - x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}})$.

4. $(x^{\frac{1}{2}} - y^{\frac{1}{2}})(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$; $(x^{\frac{2}{3}} + y^{\frac{2}{3}})(x^{\frac{4}{3}} - x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}})$.

5. $(x^2 + \sqrt{(xy)} + y^2)(x^2 - \sqrt{(xy)} + y^2)$;
$(x + 2\sqrt{(xy)} + y)(x - 2\sqrt{(xy)} + y)$.

6. $\{(2a)^{\frac{1}{2}} + b^{\frac{1}{2}}\}\{(2a)^{\frac{3}{2}} - (2a)b^{\frac{1}{2}} + (2a)^{\frac{1}{2}}b - b^{\frac{3}{2}}\}$.

7. $(x^a - 2x^{-a} - 7)(5 - 3x^{-a} + 2x^a)$; $(2x^{\frac{m}{n}} - 3y^{\frac{p}{q}})(2x^{\frac{m}{n}} + 3y^{\frac{p}{q}})$.

8. $(a^{\frac{3}{2}} - a^{\frac{2}{3}} + a^{-\frac{1}{3}} - a^{-\frac{3}{2}})(a^{\frac{3}{2}} + a^{\frac{2}{3}} - a^{-\frac{1}{3}} - a^{-\frac{3}{2}})$,

$a^3 + a^{-3} - 2 - a^{\frac{4}{3}} - a^{-\frac{2}{3}} + 2a^{3}$

9. $(x - 4y^{\frac{2}{3}})(x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + 4y^{\frac{2}{3}})(x - 2x^{\frac{1}{2}}y^{\frac{1}{2}} + 4y^{\frac{2}{3}})$.

$x^3 + 4x^2y^{\frac{2}{3}} - 4x^2y - 16xy^{\frac{4}{3}} + 16xy^{\frac{5}{3}} - 64$

10. $(x^n + x^{\frac{1}{2}n} + 1)(x^{-n} + x^{-\frac{1}{2}n} + 1)$;
$(a^{\frac{1}{3}}x^{\frac{1}{3}} - b)(a^{\frac{1}{3}}x^{\frac{1}{3}} + b)(a^{\frac{2}{3}}x^{\frac{2}{3}} + b^2)$.

$3 + 2x^{-\frac{n}{2}} + 2x^{\frac{n}{2}} + x^{-n} + x^n$; $a^{\frac{4}{3}}x^{\frac{4}{3}} -$

Divide :—

11. $x^{\frac{2}{3}} - y^{\frac{2}{3}}$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}}$; $x^{\frac{4}{3}} - y^{\frac{4}{3}}$ by $x^{\frac{1}{3}} - y^{\frac{1}{3}}$. $x + y + x^{\frac{1}{3}}y^{\frac{2}{3}} + x$

$x^{\frac{1}{3}} - y^{\frac{1}{3}}$

12. $x + y$ by $x^{\frac{1}{5}} + y^{\frac{1}{5}}$; $16a^{-3} + 6a^{-2} + 5a^{-1} - 6$ by $2a^{-1} - 1$. $8a^{-2} + 7$

$x^{\frac{4}{5}} - x^{\frac{3}{5}}y^{\frac{1}{5}} + x^{\frac{2}{5}}y^{\frac{2}{5}} - x^{\frac{1}{5}}y^{\frac{3}{5}} + y^{\frac{4}{5}}$

13. $5b^{\frac{2}{3}} - 6b^{\frac{1}{3}} - 4b^{-\frac{2}{3}} - 4b^{-\frac{1}{3}} - 5$ by $b^{\frac{1}{6}} - 2b^{-\frac{1}{6}}$.

$5b^{\frac{1}{2}} + 4b^{\frac{1}{6}} + 3b^{-\frac{1}{6}} + 2b^{-\frac{1}{2}}$

$x^{-5/3} + x^{-4/3}y^{-1/3} + x^{-1}y^{-2/3} + x^{-2/3}y^{-1} + x^{-1/3}y^{-4/3} + y^{-5/3}$

14. $x^{-2} - y^{-2}$ by $x^{-\frac{1}{3}} - y^{-\frac{1}{3}}$; $a^{-3} - 64b^2$, by $a^{-\frac{1}{2}} + 2b^{\frac{1}{3}}$.

$a^{-5/2} - 2a^{-2}b^{1/3} + 4a^{-3/2}b^{2/3} - 8a^{-1}b + 16a^{-1/2}b^{\frac{3}{3}}$

15. Square $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} + 3 - 2x^{-\frac{1}{3}} + x^{-\frac{2}{3}}$.

$x^{4/3} - 4x + 10x^{2/3} - 16x^{1/3} + 19 - 16x^{-1/3} + 10x^{-2/3} - 4x^{-1} + x^{-4/3}$

16. Resolve $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$: $x^{\frac{4}{3}} + 4y^{-\frac{4}{3}}$.

17. Resolve $x - x^{\frac{1}{2}} - 56$: $9x^{\frac{3}{2}} + 3x^{\frac{3}{4}}y^{\frac{1}{2}} - 2y$.

18. Resolve $x^2 - 1 + x^{\frac{1}{2}} - x^{\frac{3}{2}}$; $6(x + y) - 13x^{\frac{1}{2}}y^{\frac{1}{2}}$.

19. Reduce $\dfrac{a^{-1} + a^{-1}b^{-3}}{b - 1 - b^{-2}}$; $\dfrac{a^{\frac{1}{4}(3m-1)}b^{-1}}{a^{\frac{1}{4}(m-3)}b^{\frac{1}{2}(m-1)}}$; $\dfrac{a + b + c - 3a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}}{(a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^2}$.

20. Find the square root of $a^2b^{-2} + 2ab^{-1} + 3 + 2a^{-1}b + a^{-2}b^2$.

21. Find the square root of $a^{\frac{4}{3}} + 4ay^{\frac{1}{5}} + 10a^{\frac{2}{3}}y^{\frac{2}{5}} + 12a^{\frac{1}{3}}y^{\frac{3}{5}} + 9y^{\frac{4}{5}}$.

(*b*)

1. What does $a^2b^2 - a^{-2}b^{-2} - a^2 - b^2 + a^{-2} + b^{-2}$ become when $ab - a^{-1}b^{-1} = 0$?

2. Show that the expression in (1)

$$= (ab - a^{-1}b^{-1})(a - a^{-1})(b - b^{-1}).$$

3. Find the square root of $a^{\frac{2}{3}} - 4a^{\frac{5}{6}} + 4a + 2a^{\frac{7}{6}} - 4a^{\frac{4}{3}} + x^{\frac{5}{3}}$.

4. Simplify $\left(\dfrac{xy}{x}\right)^{\frac{1}{2}} \cdot \left(\dfrac{bx}{y^2}\right)^{\frac{1}{3}} \cdot \left(\dfrac{y^3}{a^3b^2}\right)^{\frac{1}{6}} \cdot \left(\dfrac{x^3y^{-3}}{x^{-3}y^3}\right)^{\frac{4}{3}}$.

5. Simplify $\left\{\left(\dfrac{1+x}{1-x}\right)^{\frac{3}{2}} - \left(\dfrac{1-x}{1+x}\right)^{\frac{3}{2}}\right\} \div \left(\dfrac{1+x}{1-x}\right)^{\frac{3}{2}}$.

6. Divide $a^2 + 2ab^{\frac{3}{2}} + b^3$ by $a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{1}{3}} + b$.

7. Multiply together $x - 1 + \sqrt{2}$, $x + 2 + \sqrt{3}$, $x - 1 - \sqrt{2}$, and $x + 2 - \sqrt{3}$.

8. Find the L. C. M. of $ax^3 - a^{\frac{1}{3}}x$, $ax^3 - 1$, and $ax^3 + 1$.

9. Find the square root of $a^2b^{-2} + \frac{1}{4}a^{-2}b^{-2} - a^{-1}b + 2ab^{-1}$.

10. If $4a = 5b = 1$ find the value of

$$\left\{\tfrac{1}{3}\left(a^{\frac{2}{3}} + a^{-\frac{1}{2}}b^{-1}\right)\right\}^{\frac{2}{3}} - \sqrt{\left[\tfrac{1}{3}\left\{1 + a^{-\frac{3}{2}} - (1 + ab^{-1})^{-\frac{1}{2}}\right\}\right]}.$$

11. Reduce to lowest terms $\dfrac{x^3 + a^2x^2 - ax - a^{\frac{3}{2}}}{x^2 - ax + a^{\frac{1}{2}}x - a^{\frac{3}{2}}}$.

12. Simplify $\{xy^{-2} \cdot \sqrt{(xy^3)} \cdot \sqrt[3]{(xy^4)} \cdot \sqrt[4]{(xy^5)} \cdot \sqrt[5]{(ab)^{-10}}\}^{\frac{1}{5}}$.

13. Show that $\sqrt{\{a^2 + \sqrt[3]{(a^4b^2)}\}} + \sqrt{\{b^2 + \sqrt[3]{(a^2b^4)}\}} = (a^{\frac{2}{3}} + b^{\frac{2}{3}})^{\frac{3}{2}}$.

14. Show that $\left(x^{\frac{m+n}{n}} - x^{\frac{n+m}{m}}\right) \div \left(x^{\frac{2m-n}{n}} - x^{\frac{2n-m}{m}}\right) = x^2 \div \left(x^{\frac{m}{n}} + x^{\frac{n}{m}}\right)$.

15. Divide $a^{7x} - a^{-7y}$ by $a^x - a^{-y}$.

16. Simplify $\left(\dfrac{a^9b^{28}c^{47}}{d^{10}e^{29}}\right)^{17} \div \left(\dfrac{a^8b^{25}c^{42}}{d^9e^{26}}\right)^{19}$.

17. Divide $x^{\frac{3}{4}} - ma^{\frac{1}{2}}x^{\frac{1}{2}} + max^{\frac{1}{4}} - a^{\frac{3}{2}}$ by $x^{\frac{1}{4}} - a^{\frac{1}{2}}$.

18. Prove $(x + x^{-1})^2 - (y + y^{-1})^2 = (xy - x^{-1}y^{-1})(xy^{-1} - x^{-1}y)$.

19. Find the H. C. F. of $4a^2x^4 + 9a^{\frac{3}{2}}x^3 + 2ax^2 - 2a^{\frac{1}{2}}x - 4$ and $3a^{\frac{3}{2}}x^3 + 5ax^2 - a^{\frac{1}{2}}x + 2$.

20. Find the L. C. M. of $ax^2 - 1$, $ax^2 + 1$, $(a^{\frac{1}{2}}x - 1)^2$, $(a^{\frac{1}{2}}x + 1)^2$, $a^{\frac{3}{2}}x^3 - 1$, $a^{\frac{3}{2}}x^3 + 1$.

21. Multiply $1 + \frac{1}{2}a^{-\frac{1}{2}}x + \frac{1}{4}a^{-1}x^2$ by $1 - \frac{1}{2}a^{-\frac{1}{2}}x + \frac{1}{8}a^{-\frac{3}{2}}x^3 - \frac{1}{16}a^{-2}x^4$.

22. Reduce $\dfrac{3ax^3 - 2a^{\frac{2}{3}}x^2 - a^{\frac{1}{3}}x}{6a^{\frac{2}{3}}x^2 - a^{\frac{1}{3}}x - 1}$; and $\dfrac{a + a^{\frac{3}{2}} - a^{\frac{5}{2}} - a^3}{a^{\frac{5}{2}} + 6a^3 + 5a^{\frac{7}{2}}}$.

23. If $x^{-1} = (a-c)(b-c)$, $y^{-1} = (a-b)(b-c)$, $z^{-1} = (a-b)(a-c)$, find the value of $x - y + z$; and of $abx - acy + bcz$.

24. Simplify $\sqrt[2n]{\{(a + b)^{m+1}\}} \cdot \sqrt{(a+b)^{\frac{n+1}{m}}} \cdot \sqrt{(a+b)^{\frac{n-1}{m}}} \cdot \sqrt[m]{(a+b)^{\frac{n-1}{2}}}$.

184. Surds.—When a required root of a quantity cannot be found *exactly*, it is *indicated*, and the quantity is called a **Surd** or *Irrational quantity*.

Thus, $\sqrt{2}$, $\sqrt[3]{9}$, $\sqrt[4]{a^3}$, $\sqrt[n]{a^m}$, $(a^2b)^{\frac{3}{4}}$, are surds.

Sometimes a *rational* quantity has the *form* of a surd.

Thus, $\sqrt[3]{8}$, $(a^mb^m)^{\frac{1}{m}}$ are surds in *form* only, for the former has a value 2, and the latter, ab.

185. From the *Law of Signs* in Multiplication, we see that :—

1°. Since an *even power* of a quantity has the same value whether the quantity is positive or negative, an *even root* of a positive quantity has the double sign ±.

Thus, if n be an integer, $(\pm a)^{2n} = a^{2n}$; $\therefore$ the $2n$th *root* of a^{2n} is $\pm a$.

2°. Since an *odd power* of a quantity has the *same sign* as the quantity itself, an *odd root of a quantity has the same sign as the quantity itself.*

Thus, if n be an integer, $(\pm a)^{2n+1} = \pm a^{2n+1}$, and therefore the $(2n+1)$th root of $\pm a^{2n+1}$ is $\pm a$.

3°. Since no *even power* of a quantity can be negative, *no even root of a negative quantity can be found.* Such roots may be *indicated*, *e. g.*, $\sqrt{-4}$, $\sqrt[4]{-3}$, $\sqrt[2n]{-1}$, etc. They are called *impossible* or *imaginary* quantities, and are used in higher Algebra.

186. The order of a surd is determined by its *root* sign (Art. 37), or fractional index, which indicates the root to be extracted.

Thus, $\sqrt{2}$, $\sqrt{(ab)}$, are surds of the *second* order, or *quadratic* surds; $\sqrt[3]{a}$, $\sqrt[3]{b}$, are of the *third* order, or *cubic* surds; $\sqrt[4]{7}$, $a^{\frac{1}{4}}$ are of the *fourth* order; $\sqrt[n]{a^m}$, or $a^{\frac{m}{n}}$, is of the nth order.

187. A Rational Quantity may be expressed as a *surd of any order.*

Thus, $2 = \sqrt{2^2} = \sqrt[3]{2^3} = \sqrt[4]{2^4} = \sqrt[n]{2^n}$; $a = (a^2)^{\frac{1}{2}} = (a^3)^{\frac{1}{3}} = (a^n)^{\frac{1}{n}}$.

Also, the *order* of a surd can be changed.

Thus, $2^{\frac{1}{2}} = 2^{\frac{2}{4}} = 2^{\frac{3}{6}}$, etc., $3^{\frac{2}{3}} = 3^{\frac{4}{6}} = 3^{\frac{8}{12}}$, etc., and generally, $a^{\frac{p}{q}} = a^{\frac{pn}{qn}}$; for $a^{\frac{pn}{q}}$ represents the nth *power* of $a^{\frac{p}{q}}$, and $a^{\frac{pn}{qn}}$ represents the nth *root* of this nth power, *i. e.*, the quantity $a^{\frac{p}{q}}$ itself.

188. A Mixed Surd is one which has a rational factor; this may be separated and written as a coefficient of the surd.

Thus, $\sqrt{18} = \sqrt{(2 \times 9)} = \sqrt{2} \times \sqrt{9} = 3\sqrt{2}$;

$$3\sqrt[3]{(a^6b^2)} = 3\sqrt[3]{a^6} \times \sqrt[3]{b^2} = 3a^2\sqrt[3]{b^2}.$$

Conversely, it is plain that a mixed surd may be changed to an *entire* surd by giving the *rational* factor a *surd-form* of the same order as the *surd factor*. Thus,

$$3\sqrt{3} = \sqrt{3^2} \times \sqrt{3} = \sqrt{(3^2 \times 3)} = \sqrt{27};$$

$$2\sqrt[3]{7} = \sqrt[3]{2^3} \times \sqrt[3]{7} = \sqrt[3]{(2^3 \times 7)} = \sqrt[3]{56};$$

$$a\sqrt[4]{b} = \sqrt[4]{a^4} \times \sqrt[4]{b} = \sqrt[4]{(a^4 b)};$$

$$ab^{\frac{1}{2}} = (a^2 b)^{\frac{1}{2}}; \quad ab^{\frac{1}{n}} = (a^n b)^{\frac{1}{n}}.$$

189. Similar or Like Surds are those that have, or can be made to have, the same *order* and the same *surd factor*.

Thus, $2\sqrt{3}$, $4\sqrt{3}$; $5\sqrt[3]{7}$, $3\sqrt[3]{7}$, are pairs of *like* surds; so $3\sqrt{3}$ $2\sqrt[4]{9}$, are *like* surds, for the latter $= 2\sqrt{\sqrt{9}} = 2\sqrt{3}$.

190. A surd is reduced to its **simplest form,** when its surd factor is reduced to the smallest possible integral number.

Thus, $\sqrt{12} = \sqrt{3} \times \sqrt{4} = 2\sqrt{3}$;

$$\sqrt[3]{256} = \sqrt[3]{4} \times \sqrt[3]{64} = 4\sqrt[3]{4};$$

$$\sqrt{(27a^3b^5)} = \sqrt{(3ab)} \times \sqrt{(9a^2b^4)} = 3ab^2\sqrt{(3ab)}.$$

191. Surds are **Compared in Magnitude** by reducing them to *entire* surds, and then to the *same surd index.*

Thus, $3\sqrt{5}$, $5\sqrt{3}$, give $\sqrt{45}$, $\sqrt{75}$ respectively; $\sqrt{5}$, $\sqrt[3]{11}$, give $5^{\frac{1}{2}}$, $11^{\frac{1}{3}}$, or $5^{\frac{3}{6}}$, $11^{\frac{2}{6}}$, or $125^{\frac{1}{6}}$, $121^{\frac{1}{6}}$; so that $\sqrt{5}$ is seen to be greater than $\sqrt[3]{11}$.

EXERCISE XC.

1. Change to the surd form :—

$$x^{\frac{2}{3}}; \quad 7x^{\frac{1}{2}}y^{\frac{3}{2}}; \quad 5x^{\frac{3}{4}}y^{\frac{7}{4}}; \quad 6x^{\frac{3}{2}}y^{\frac{2}{3}}; \quad a^{\frac{3}{n}}b^{\frac{4}{n}}.$$

2. Express with exponents whose numerators are each unity :—

$$3^{\frac{3}{4}}; \quad 8^{\frac{3}{5}}; \quad 3^{-\frac{2}{3}}; \quad (\tfrac{2}{3})^{-\frac{1}{4}}; \quad (\tfrac{1}{2})^{-\frac{3}{5}}.$$

3. Express with exponents $\frac{1}{3}$ and $-\frac{1}{4}$:—

$$-\tfrac{2}{3}b^2; \quad \tfrac{3}{4}a^2b^3; \quad 4^{-3}; \quad (3\tfrac{1}{3})^{-1}; \quad ab^{-1}c^{-3}.$$

Reduce to entire surds :—

4. $3\sqrt{4}$; $5\sqrt[3]{2}$; $\frac{1}{2}\sqrt{2}$; $2\sqrt{\frac{3}{2}}$; $3\sqrt[3]{3\frac{1}{3}}$; $2\sqrt[3]{2}$; $\frac{1}{3}\sqrt[4]{27}$; $a\sqrt[3]{a}$; $b\sqrt[4]{(a^2b)}$.

5. $a\sqrt{\frac{b}{a}}$; $\frac{a}{b}\sqrt{\frac{b}{a}}$; $\frac{a}{b}\sqrt[3]{\frac{b}{a}}$; $3ax\sqrt{\frac{2a}{3x}}$; $\frac{2a}{3b}\sqrt[3]{\frac{2b}{3a}}$; $\frac{4a}{3}\sqrt[3]{\frac{9}{4a^2}}$;
$(a+b)\sqrt{\left(\frac{a-b}{a+b}\right)}$.

6. $a\sqrt[n]{b}$; $a\sqrt[n]{a}$; $(a-x)\sqrt[n]{(a+x)^{n+1}}$;
$\frac{a+b}{c}\sqrt{\frac{a+b}{c}}$; $\frac{x-3}{x+4}\sqrt[3]{\frac{x+4}{x-3}}$.

Reduce to the simplest form :—

7. $\sqrt{90}$; $\sqrt{125}$; $\sqrt[3]{135}$; $3\sqrt{54}$; $3\sqrt[3]{432}$; $3\sqrt[3]{\frac{3}{2}}$; $5\sqrt[3]{3\frac{3}{8}}$.

8. $32^{\frac{2}{3}}$; $72^{\frac{3}{5}}$; $80^{\frac{1}{4}}$; $48^{\frac{1}{4}}$; $90{,}000^{\frac{1}{4}}$;
$\sqrt[5]{32}$; $\sqrt[3]{144}$; $1728^{\frac{1}{3}}$; $\sqrt[n]{(a^n b^{n+1})}$.

9. $(1\frac{1}{8})^{-\frac{1}{2}}$; $(20\frac{1}{4})^{-\frac{3}{4}}$; $(30\frac{3}{8})^{-\frac{3}{5}}$; $(a^4y^8)^{\frac{1}{4}}$; $\sqrt[n]{a^{n+m}}$;
$a\sqrt[3]{(a^4x^5-a^3x^3)}$.

10. $5\sqrt[3]{4\frac{1}{20}}$; $\frac{5}{2}\sqrt[4]{9\frac{3}{5}}$; $\sqrt{(ax^2+10ax+25a)}$;
$\sqrt[3]{\{(x+y)^2(x^2-y^2)\}}$.

11. $\{(x^2-a^2)^2(x-a)^2\}^{\frac{1}{3}}$; $(x^3+2x^2y+xy^2)^{\frac{1}{2}}$;
$\sqrt[3]{(8a^4b-24a^3b^2+24a^2b^3-8ab^4)}$.

12. Show that $2\sqrt{75}$, $\frac{1}{2}\sqrt{147}$, $\sqrt{\frac{4}{75}}$, $\sqrt[4]{\frac{9}{16}}$, and $144^{-\frac{1}{4}}$ are similar surds.

Reduce the following to the same surd indices, so as to compare their magnitudes :—

13. 2, $\sqrt{3}$; 2, $\sqrt[3]{6}$; 10, $\sqrt[4]{1000}$; $\sqrt[5]{33}$, 2; $2\sqrt[3]{10}$, $\sqrt[3]{50}$;
a^2, $\sqrt{a^3}$; $\sqrt[6]{a}$, $\sqrt[8]{b}$; $\sqrt[5]{a^7}$, $\sqrt[7]{a^5}$.

14. Write in the order of magnitude: $2\sqrt[3]{3}$, $3\sqrt{2}$, $\frac{5}{2}\sqrt[4]{1}$.

192. If surds are *similar*, they can be **added** or **subtracted** by *adding or subtracting their rational coefficients.* If they are *dissimilar*, we merely connect them with their proper signs as in the case of ordinary unlike quantities.

Thus, $3\sqrt{8}+\sqrt{50}-\sqrt{98} = 2\cdot3\sqrt{2}+5\sqrt{2}-7\sqrt{2}$
$= (11-7)\sqrt{2} = 4\sqrt{2}$.

193. To Multiply Surds we *reduce them to the same surd index*, and then *multiply the product of the rational parts into the product of the surd factors.*

Thus, $3\sqrt{2} \times 2\sqrt{18} = 3 \times 2\sqrt{(2 \times 18)} = 6 \times 6 = 36$.

Also. $2\sqrt[3]{2} \times 2\sqrt{2} = 2 \cdot 2^{\frac{1}{3}} \times 2 \cdot 2^{\frac{1}{2}} = 2 \cdot 2^{\frac{2}{6}} \times 2 \cdot 2^{\frac{3}{6}} = 4 \cdot 2^{\frac{2}{6}} \cdot 2^{\frac{3}{6}}$ $= 4 \cdot 4^{\frac{1}{6}} \cdot 8^{\frac{1}{6}} = 4 \cdot 32^{\frac{1}{6}} = 4\sqrt[6]{32}$.

194. Polynomial Surds are multiplied, term into term, as in the case of ordinary Multiplication.

Thus, $(2 + 3\sqrt{2})(3 - 2\sqrt{2}) = 6 - 4\sqrt{2} + 9\sqrt{2} - 6\sqrt{(2 \cdot 2)}$ $= 5\sqrt{2} - 6$.

195. To Divide a Surd, monomial, or polynomial, **by a monomial surd**, we *reduce to the same surd index*, when necessary, and divide every term of the dividend by the divisor.

Thus, $(8\sqrt{2} + 12\sqrt{3}) \div 2\sqrt{6} = 4\sqrt{\tfrac{2}{6}} + 6\sqrt{\tfrac{3}{6}} = 4\sqrt{\tfrac{1}{3}} + 6\sqrt{\tfrac{1}{2}}$ $= \tfrac{4}{3}\sqrt{3} + 3\sqrt{2}$.

And $(2\sqrt{3} + 6\sqrt[3]{2}) \div \sqrt{6} = 2\sqrt{\tfrac{3}{6}} + 6 \cdot 2^{\frac{1}{3}} \div 6^{\frac{1}{2}} = \sqrt{2} + 6 \cdot 2^{\frac{2}{6}} \div 6^{\frac{3}{6}}$ $= \sqrt{2} + 6\sqrt[6]{\tfrac{4}{216}} = \sqrt{2} + \sqrt[6]{864}$.

196. If the divisor is a **Binomial Quadratic Surd** of the form $\sqrt{a} \pm \sqrt{b}$, we *write the quotient in the form of a fraction* and then multiply both terms by $\sqrt{a} \mp \sqrt{b}$, to *make the denominator* (divisor) *rational:* this is called *rationalizing the denominator.*

Thus, $(\sqrt{5} + \sqrt{2}) \div (\sqrt{5} - \sqrt{2}) = \dfrac{\sqrt{5} + \sqrt{2}}{\sqrt{5} - \sqrt{2}} = \dfrac{(\sqrt{5} + \sqrt{2})^2}{5 - 2}$ $= \dfrac{7 + 2\sqrt{10}}{3}$.

N. B.—If the divisor is a surd of a *higher* degree than the second, the work will be more difficult; this case will not be considered here.

Ex. 1. Combine $5\sqrt[3]{80} + 2\sqrt[3]{270} + \sqrt[3]{640}$.

We have, by *simplifying*,

$2 \cdot 5\sqrt[3]{10} + 3 \cdot 2\sqrt[3]{10} + 4\sqrt[3]{10} = (10 + 6 + 4)\sqrt[3]{10} = 20\sqrt[3]{10}$.

Ex. 2. $3\sqrt{147} - \tfrac{7}{3}\sqrt{\tfrac{1}{3}} - \sqrt{\tfrac{1}{27}}$.

We have $7 \cdot 3\sqrt{3} - \tfrac{1}{3} \cdot \tfrac{7}{3}\sqrt{3} - \tfrac{1}{9}\sqrt{3} = (21 - \tfrac{8}{9})\sqrt{3} = 20\tfrac{1}{9}\sqrt{3}$.

Ex. 3. Multiply $a\sqrt{(1 - b^2)} + b\sqrt{(1 - a^2)}$ by $a\sqrt{(1 - b^2)}$ $- b\sqrt{(1 - a^2)}$.

Applying formula (B), we have $a^2(1 - b^2) - b^2(1 - a^2) = a^2 - b^2$.

Ex. 4. Multiply together $3\sqrt{8}$, $2\sqrt[3]{6}$, and $3\sqrt[4]{54}$.

The product of the coefficients is $3\cdot 2\cdot 3 = 18$.

The product of the surd factors is

$$\sqrt{8}\cdot\sqrt[3]{6}\cdot\sqrt[4]{54} = 8^{\frac{1}{2}}\cdot 6^{\frac{1}{3}}\cdot 54^{\frac{1}{4}} = 8^{\frac{6}{12}}\cdot 6^{\frac{4}{12}}\cdot 54^{\frac{3}{12}} = (8^6\cdot 6^4\cdot 54^3)^{\frac{1}{12}}$$

$$= (2^{18}\times 3^4\cdot 2^4\times 2^3\cdot 3^9)^{\frac{1}{12}} = (3^{12}\cdot 2^{24}\cdot 2\cdot 3)^{\frac{1}{12}}$$

$$= 3\cdot 2^2\,(3\cdot 2)^{\frac{1}{12}} = 12\sqrt[12]{6}.$$

Now multiply this by 18, the product of the coefficients, and we have

$$18\times 12\sqrt[12]{6} = 216\sqrt[12]{6}.$$

This example shows that sometimes labour may be saved by *factoring* the quantities to be multiplied.

Ex. 5. Divide $8 - 5\sqrt{2}$ by $3 - 2\sqrt{2}$.

We have $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}} = \dfrac{(8-5\sqrt{2})(3+2\sqrt{2})}{9-4\cdot 2} = 4+\sqrt{2}.$

Ex. 6. Simplify $\dfrac{x+\sqrt{(x^2-1)}}{x-\sqrt{(x^2-1)}} - \dfrac{x-\sqrt{(x^2-1)}}{x+\sqrt{(x^2-1)}}.$

The expression

$$= \frac{\{x+\sqrt{(x^2-1)}\}^2 - \{x-\sqrt{(x^2-1)}\}^2}{x^2-(x^2-1)} = \frac{4x\sqrt{(x^2-1)}}{1} = 4x\sqrt{(x^2-1)}.$$

Ex. 7. Simplify $\dfrac{c\sqrt{(ab)}-ac}{bc-c\sqrt{(ab)}}.$

$$\text{Expression} = \frac{c\{\sqrt{(ab)}-a\}}{c\{b-\sqrt{(ab)}\}} = \frac{\sqrt{a}\,(\sqrt{b}-\sqrt{a})}{\sqrt{b}\,(\sqrt{b}-\sqrt{a})} = \sqrt{\frac{a}{b}}.$$

Ex. 8. Simplify $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}.$

$$\text{The expression} = \frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{(\sqrt{2}+\sqrt{3})^2-5} = \frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2\sqrt{6}}$$

$$= \tfrac{1}{2}\sqrt{\tfrac{1}{3}} + \tfrac{1}{2}\sqrt{\tfrac{1}{2}} + \tfrac{1}{2}\sqrt{\tfrac{5}{6}} = \tfrac{1}{6}\sqrt{3} + \tfrac{1}{4}\sqrt{2} + \tfrac{1}{12}\sqrt{30}.$$

EXERCISE XCI.

1. Simplify: $3\sqrt{128} - 5\sqrt{50} - \sqrt{72} + 3\sqrt{18}$; $2\sqrt{2}$
   $2\sqrt[3]{40} - \frac{1}{2}\sqrt[3]{320} + \sqrt[3]{1080}$. $8\sqrt[3]{5}$

2. Simplify: $4\sqrt{\frac{3}{4}} + \frac{1}{2}\sqrt{12} - 5\sqrt{27} - 2\sqrt{\frac{3}{16}}$; $-\frac{25}{2}\sqrt{3}$
   $3\sqrt[3]{72} - 3\sqrt[3]{\frac{1}{3}} + 5\sqrt[3]{21\frac{1}{3}}$.

3. Multiply: $3\sqrt{8}$ by $5\sqrt{6}$; $2\sqrt{15}$ by $4\sqrt{20}$; $60\sqrt{3}$; $80\sqrt{3}$;
   $2\sqrt[3]{4}$ by $2\sqrt[3]{54}$. 24

4. Multiply: $3\sqrt{3} + 2\sqrt{2}$ by $2\sqrt{3} - 3\sqrt{2}$; $6 - 5\sqrt{6}$
   $2\sqrt{15} - \sqrt{6}$ by $\sqrt{5} + 2\sqrt{2}$. $6\sqrt{3} + 3\sqrt{30}$

5. Find the continued product of $4 + 2\sqrt{2}$, $4 - 2\sqrt{2}$, $1 + \sqrt{5}$, and $1 - \sqrt{5}$. -32

6. Divide: $2\sqrt{3} + 3\sqrt{2} + 6\sqrt{30}$ by $3\sqrt{6}$;
   $2\sqrt{3} + 3\sqrt[3]{2} + \sqrt[4]{30}$ by $3\sqrt{2}$. $\frac{1}{3}\sqrt{6} + \frac{1}{2}\sqrt[6]{2} + \frac{1}{6}\sqrt[4]{30}$

7. Divide: 3 by $\sqrt{7} - \sqrt{3}$; $3 + \sqrt{5}$ by $3 - \sqrt{5}$.

8. Rationalize the denominators of

$$\frac{3 - \sqrt{5}}{3 + \sqrt{5}};\quad \frac{3 - 5\sqrt{2}}{4 + 2\sqrt{2}};\quad \frac{4\sqrt{7} + 3\sqrt{2}}{5\sqrt{2} + 2\sqrt{7}};\quad \frac{a + \sqrt{(a^2 - 1)}}{a - \sqrt{(a^2 - 1)}}.$$

9. Find the continued product of $2\sqrt{24}$, $3\sqrt[4]{18}$, and $4\sqrt[6]{24}$.

10. Square $\sqrt{x} + \sqrt{y} - \sqrt{z}$; $3x + 2\sqrt{(x^2 + 1)}$.

11. Simplify: $\dfrac{x - a}{\sqrt{x} + \sqrt{a}}$; $\dfrac{x + a}{\sqrt[3]{x} + \sqrt[3]{a}}$; $\dfrac{a^2 + ab + b^2}{a + \sqrt{(ab)} + b}$;
    $\dfrac{3 - \sqrt{2}}{3 + \sqrt[4]{2}}$.

12. If $\sqrt{(ax)} + \sqrt{(by)} + \sqrt{(cz)} = 0$, then
    $a^2x^2 + b^2y^2 + c^2z^2 = 2(abxy + acxz + bcyz)$.

13. Prove $2\{x + 2 + \sqrt{(x^2 - 4)}\} \div \{x + 2 - \sqrt{(x^2 - 4)}\}$
    $= x + \sqrt{(x^2 - 4)}$.

14. Find the value of $7\sqrt{5} \div (\sqrt{11} + \sqrt{3})$ to four decimal places.

15. Find the value of $(3 + 2\sqrt{7}) \div (5\sqrt{12} - 6\sqrt{6})$ to three decimal places.

16. Simplify; $\dfrac{\sqrt{(3+\sqrt{5})} - \sqrt{(5-\sqrt{5})}}{\sqrt{(3+\sqrt{5})} + \sqrt{(5-\sqrt{5})}}$; $\dfrac{\sqrt{(a+x)} + \sqrt{(a-x)}}{\sqrt{(a+x)} - \sqrt{(a-x)}}$.

$(\sqrt{5}+1)\{4 - \sqrt{10+2\sqrt{5}}\} \div 4$ $\{a + \sqrt{a^2-x^2}\} \div x$

17. Simplify: $\dfrac{x + \sqrt{(x^2-1)}}{x - \sqrt{(x^2-1)}} - \dfrac{x - \sqrt{(x^2-1)}}{x + \sqrt{(x^2-1)}}$; $4x\sqrt{x^2-1}$

$$\frac{1}{4(1+\sqrt{x})} + \frac{1}{4(1-\sqrt{x})} + \frac{1}{2(1+x)}. \quad \frac{1}{1-x^2}$$

18. Simplify $\dfrac{\sqrt{(x^2+a^2)} + \sqrt{(x^2-a^2)}}{\sqrt{(x^2+a^2)} - \sqrt{(x^2-a^2)}} + \dfrac{\sqrt{(x^2+a^2)} - \sqrt{(x^2-a^2)}}{\sqrt{(x^2+a^2)} + \sqrt{(x^2-a^2)}}$. $\dfrac{2x}{a}$

19. Find the value of $\dfrac{\sqrt{(1+x)} + \sqrt{(1-x)}}{\sqrt{(1+x)} - \sqrt{(1-x)}}$, when $x = \dfrac{2a}{a^2+1}$. a

20. Simplify

$$\frac{\sqrt{(a+x)} - \sqrt{(a-x)}}{a + x + \sqrt{(a^2-x^2)}} \times \frac{\sqrt{(a^2-x^2)}}{\sqrt{a} + \sqrt{x}} \times \frac{x}{a - \sqrt{(a^2-x^2)}}. \quad \frac{\sqrt{a-}}{\sqrt{a+}}$$

21. Simplify

$$\frac{6xy}{m^{\frac{1}{2}} + n^{\frac{1}{2}}} \div \left[\frac{3(m^{\frac{1}{2}} - n^{\frac{1}{2}})\,x^{\frac{1}{2}}}{7(r^{\frac{1}{2}} + s^{\frac{1}{2}})} \div \left\{\frac{4(r^{\frac{1}{2}} - s^{\frac{1}{2}})}{21x^{\frac{1}{2}}y} \div \frac{r-s}{4(m-n)}\right\}\right]. \quad 10$$

197. If $a + \sqrt{b} = x + \sqrt{y}$, where a and b are *rational*, and $\sqrt{b}$ and $\sqrt{y}$ are *irrational*, then $a = x$ and $b = y$; *i. e.*, the rational parts are equal, and, also, the irrational parts. For, we have
$$a - x + \sqrt{b} = \sqrt{y},$$
and, squaring, $(a-x)^2 + 2(a-x)\sqrt{b} + b = y$,

or $$2(a-x)\sqrt{b} = y - b - (a-x)^2,$$
in which the *coefficient of* $\sqrt{b}$ *must vanish*,
$$i.\,e.,\ a - x = 0, \quad \text{or} \quad a = x;$$
otherwise we should have an *irrational* quantity equal to a *rational* one, which is impossible.

198. This proposition enables us to find in a simple form, where possible, the **square root of a binomial surd,** *one of whose terms is rational and the other a quadratic surd.*

Thus, to find the square root of $a + \sqrt{b}$, assume

$$\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y};$$

then, squaring both sides.

$$\therefore \quad a + \sqrt{b} = x + y + 2\sqrt{xy};$$

whence by the last Art., $x + y = a$, (1)

and $\sqrt{b} = 2\sqrt{xy}$,

or $4xy = b$. (2)

Solving (1) and (2) we get $x = \frac{1}{2}\{a + \sqrt{(a^2 - b)}\}$,

$$y = \tfrac{1}{2}\{a - \sqrt{(a^2 - b)}\};$$

$$\therefore \sqrt{(a + \sqrt{b})} = \sqrt{\tfrac{1}{2}\{a + \sqrt{(a^2 - b)}\}} + \sqrt{\tfrac{1}{2}\{a - \sqrt{(a^2 - b)}\}}.$$

It is plain, however, that unless $a^2 - b$ is a perfect square, the right side of this identity is more complex than the left, and the *method therefore fails.*

Ex. 1. Find the square root of $7 + 2\sqrt{10}$.

Assume $\sqrt{(7 + 2\sqrt{10})} = \sqrt{x} + \sqrt{y}$;

$$\therefore \quad 7 + 2\sqrt{10} = x + y + 2\sqrt{xy};$$

$$\therefore \quad x + y = 7. \qquad (1)$$

$$xy = 10. \qquad (2)$$

$(1)^2 - 4\,(2)$ gives $49 - 40 = (x - y)^2$,

or $3 = x - y$. (3)

(1) + (3) gives $x = 5$,

(1) − (3) " $y = 2$.

$\therefore$ the required root is $\sqrt{2} + \sqrt{5} = \sqrt{2} + \sqrt{5}$.

199. When the root can be extracted, it can be found, sometimes, by *inspection*, by remembering that, in the general case, $x + y = a$, and $xy = b$. Thus, in Ex. 1: find two numbers whose sum is 7 and product 10; they are, at once, 5 and 2.

Ex. 2. $\sqrt{(4 + 2\sqrt{3})}$. Here we find two numbers whose *sum* is 4 and *product* 3; they are 3 and 1; hence the required root

$$= \sqrt{1} + \sqrt{3} = 1 + \sqrt{3}.$$

Ex. 3. Find the square root of $11+6\sqrt{2}$. Expression $=11+2\sqrt{18}$; the two numbers whose *sum* is 11 and *product* 18 are 9 and 2; hence the required root is $\sqrt{9}+\sqrt{2}=3+\sqrt{2}$.

Ex. 4. Find the square root of $3\sqrt{6}+2\sqrt{12}$. The expression

$$=\sqrt{6}\,\{3+2\sqrt{2}\},$$

and the square root of this

$$=\sqrt[4]{6}\times\sqrt{(3+2\sqrt{2})}=\sqrt[4]{6}\,(1+\sqrt{2})=\sqrt[4]{6}+\sqrt[4]{24}.$$

Ex. 5. Find the square root of $2a+2\sqrt{(a^2-b^2)}$. The two quantities whose sum is $2a$ and product a^2-b^2, are $a+b$, $a-b$: hence the required root is $\sqrt{(a+b)}+\sqrt{(a-b)}$.

EXERCISE XCII.

(*a*)

Find the square roots of

1. $5+2\sqrt{6}$.
2. $13+2\sqrt{22}$.
3. $16-4\sqrt{15}$.
4. $6+4\sqrt{2}$.
5. $16+2\sqrt{55}$.
6. $7+2\sqrt{10}$.
7. $7+2\sqrt{6}$.
8. $9+4\sqrt{5}$.
9. $57+12\sqrt{15}$.
10. $\sqrt{175}-\sqrt{147}$.
11. $10-2\sqrt{21}$.
12. $13-6\sqrt{2}$.
13. $29-12\sqrt{5}$.
14. $140-6\sqrt{451}$.
15. $9-2\sqrt{14}$.

(*b*)

Find the square roots of

1. $\sqrt{27}+2\sqrt{6}$.
2. $3\sqrt{5}+2\sqrt{10}$.
3. $5\sqrt{2}-2\sqrt{12}$.
4. $31+\sqrt{600}$.
5. $100-2\sqrt{2499}$.
6. $36+2\sqrt{323}$.
7. $3\sqrt{6}+2\sqrt{12}$.
8. $5\sqrt{2}-2\sqrt{12}$.
9. $117+36\sqrt{10}$.
10. $\frac{9}{8}-\sqrt{\frac{9}{8}}$.
11. $30\frac{1}{2}-10\sqrt{(.6)}$.
12. $2+\frac{1}{2}\sqrt{18}$.
13. $\sqrt{27}+2\sqrt{6}$.
14. $ab-2ab\sqrt{(ab-b^2)}$.
15. $1+\sqrt{(1-x^2)}$.
16. $9m+25n-30\sqrt{mn}$.
17. $m^2+2n\sqrt{(m^2-n^2)}$.
18. $2x+2\sqrt{(x^2-y^2)}$.
19. $x+y+z+2\sqrt{z(x+y)}$.
20. $2+2(1-x)\sqrt{(1+2x-x^2)}$.

Find the fourth roots of

21. $17+12\sqrt{2}$.
22. $248+32\sqrt{6}$.
23. $56-24\sqrt{5}$.
24. $49\frac{1}{16}+11\frac{1}{2}\sqrt{15}$.
25. $49-20\sqrt{6}$.
26. $6\sqrt{5}+14$.

200. Equations Involving Surds.—We shall now give a few examples, illustrating some of the artifices employed in solving the simpler forms of equations which *involve surds*.

Ex. 1. $$\sqrt{x} - \sqrt{(x-8)} = \frac{2}{\sqrt{(x-8)}}.$$

Clear of fractions

$$\therefore \quad \sqrt{(x^2 - 8x)} - (x - 8) = 2.$$

Transpose, etc., $\therefore \quad \sqrt{(x^2 - 8x)} = x - 6.$

Square both sides, $\therefore \quad x^2 - 8x = x^2 - 12x + 36,$

or $4x = 36,$ and $\therefore \; x = 9.$

Ex. 2. $x + b - \sqrt{(x^2 - ax + b^2)} = a.$

$$\therefore \quad x + b - a = \sqrt{(x^2 - ax + b^2)}.$$

Square, $\therefore \quad (x + b - a)^2 = x^2 - ax + b^2;$

$$\therefore \quad (2b - a)\,x = a\,(2b - a),$$

and $x = a.$

In these examples it is well to transpose so as to get, on squaring, results as simple as possible.

Ex. 3. $$x^2 + 4\sqrt{(x^2 - 9)} = 41.$$

Subtract 9 from each side,

$$\therefore \quad x^2 - 9 + 4\sqrt{(x^2 - 9)} = 32;$$

treat $\sqrt{(x^2 - 9)}$ as the *unknown quantity* $= y$, suppose,

that is, $x^2 - 9 = y^2,$

and the equation becomes $y^2 + 4y = 32;$

$$\therefore \quad y = 4, \text{ or } -8;$$

$$\therefore \quad \sqrt{(x^2 - 9)} = 4,$$

or $x^2 - 9 = 16,$

and $x = \pm 5;$

the other value of y gives no solution.

Ex. 4. $x^2 - 17x - 12\sqrt{(x^2 - 17x + 61)} + 72 = 0;$

Take $\sqrt{(x^2 - 17x + 61)}$ as the *unknown* quantity $= y$ suppose, and we have as in the last example,

$$y^2 - 12y + 11 = 0,$$

or $$(y-11)(y-1)=0,$$

$$\therefore\ \ y=11,\ \text{ or } 1;$$

$$\therefore\ \ \sqrt{(x^2-17x+61)}=1;$$

square and transpose, $\therefore\ \ x^2-17x+60=0,$

or $$(x-12)(x-5)=0;$$

$$\therefore\ \ x=5,\ \text{ or } 12.$$

Similarly, the other value of y gives $x=20$, or -3.

Exs. 3 and 4 show that it is sometimes convenient to *take the quantity under the root sign as the unknown quantity.*

Ex. 5. $$\frac{\sqrt{(7x^2+4)}+2\sqrt{(3x-1)}}{\sqrt{(7x^2+4)}-2\sqrt{(3x-1)}}=7.$$

Apply Art. 133, $\therefore\ \dfrac{\sqrt{(7x^2+4)}}{2\sqrt{(3x-1)}}=\dfrac{7+1}{7-1}=\dfrac{8}{6}=\dfrac{4}{3};$

multiply by 2 and square, $\therefore\ \dfrac{7x^2+4}{3x-1}=\dfrac{64}{9},$

and clearing of fractions, etc. $63x^2-192x+100=0;$

i. e., $(21x-50)(3x-2)=0,$

$$\therefore\ \ x=2\tfrac{8}{21},\ \frac{2}{3}.$$

N. B.—The student should note that in solving surd equations, values of x may sometimes be found which will *not satisfy* the equation. Thus, on solving the equation

$$4\sqrt{(1+x)}-3\sqrt{(1-x)}=5,\quad \text{we get}\quad x=\pm\frac{24}{25},$$

of which *only* the positive value will satisfy the equation. The reason is that in *clearing of radicals*, we really introduce a *rationalizing factor involving* x, *i. e.*, we combine a new equation with the given one; thus, $-\frac{24}{25}$ will *not* satisfy the given equation, but *will* satisfy the conjoint equation,

$$4\sqrt{(1+x)}+3\sqrt{(1-x)}=5.$$

It sometimes happens that some of the *conjoint* equations thus introduced have no solution. Let the student solve, *e. g.*,

$$\sqrt{(x+4)}-\sqrt{(x-4)}=4,$$

and he will find factors (involving x) which do not vanish for any value of x.

EXERCISE XCIII.

(*a*)

Find x in the following :—

1. $\sqrt{2x} = a.$
2. $\sqrt[3]{x^2} = 4.$
3. $2\sqrt{(x^2 + 4)} = 6.$
4. $\sqrt{(x + 4)} + x = 26.$
5. $\sqrt{2(x - 1)} = 4.$
6. $x^{\frac{5}{3}} = 4.$
7. $\sqrt[m]{x} = a.$
8. $\sqrt{x} + \dfrac{\sqrt{(3x + 1)}}{\sqrt{x}} = 5\tfrac{3}{4}.$
9. $\sqrt{(x^2 - 1)} = (x + 1).$
10. $\sqrt{(x^2 - a^2)} = x - a.$
11. $\sqrt{x} + \dfrac{1}{\sqrt{x}} = 5\tfrac{1}{6}.$
12. $x - \sqrt{(x^2 + 7)} + 1 = 0.$
13. $\sqrt{(x + 16)} = 2 + \sqrt{x}.$
14. $x + 2\sqrt{(x^2 + x + 5)} = 14.$
15. $\sqrt{(x + 5)} + 2x = 11.$
16. $\sqrt[3]{(2x + 3)} = 9.$
17. $\sqrt{x} + 5 = 8.$
18. $\sqrt[4]{x^5} = 2\sqrt[4]{2}.$
19. $\sqrt{(3x + 3)} + \sqrt{(x + 2)} = 5.$
20. $\sqrt{(4x+5)} - 2\sqrt{(x-1)} = 1.$
21. $\sqrt{(x + 3)} + \dfrac{\sqrt{(x - 2)}}{\sqrt{(x + 3)}} = 3\tfrac{2}{5}.$
22. $2x - 7 = 5\sqrt{(2x - 7)}.$
23. $x^{\frac{2}{5}} + x^{\frac{1}{5}} - 6 = 0.$
24. $\sqrt[3]{(ax - b)} - \sqrt[3]{(bx + a)} = 0.$
25. $\sqrt{5(x + 16)} - 5\sqrt{5} = 0.$
26. $\sqrt{(4 + x)} + \sqrt{(x + 1)} - 2 = 0.$

(*b*)

1. $\sqrt{(bx + x^2)} = 1 + x.$
2. $b + x - \sqrt{(2bx + x^2)} = a.$
3. $\sqrt{(x-1)} = \sqrt{\{x - \sqrt{(2 + x)}\}}.$
4. $\dfrac{2\sqrt{x} - 1}{2\sqrt{x} + 1\frac{1}{3}} = \dfrac{\sqrt{x} - 2}{\sqrt{x} - 1\frac{1}{3}}.$
5. $\sqrt{(x-a)} - \sqrt{\{x + (b + x)\}} = 0.$
6. $\sqrt{(x+18)} + \sqrt{(x-10)} - 14 = 0.$
7. $x - 6\sqrt{(x + 3)} + 12 = 0.$
8. $\sqrt{(1+x)} + \sqrt{x} = \dfrac{2}{\sqrt{(1+x)}}.$
9. $\dfrac{1}{1 - x} + \dfrac{1}{\sqrt{(x + 1)}} + \dfrac{1}{\sqrt{(x - 1)}} = 0.$
10. $\dfrac{2x}{\sqrt{(x - 2)} + 1} = 2\sqrt{(x - 2)} - 1.$
11. $\sqrt{x} + \sqrt{(4a + x)} = 2\sqrt{(b + x)}.$

12. $\sqrt{(12x-5)} + \sqrt{(3x-1)} = \sqrt{(27x-2)}$.

13. $x^2 - 33 + 6\sqrt{(x^2+7)} = 0$.

14. $x^2 - 2x + 6\sqrt{(x^2-2x+5)} - 11 = 0$.

15. $6\sqrt{(x^2-5x+10)} - (x^2-5x+18) = 0$.

16. $\sqrt{(a^2+x^2)} + \sqrt{(a^2-x^2)} = b$.

(*c*)

1. $12\sqrt{(x^2-10x+40)} = x^2 - 10x + 72$.

2. $\dfrac{2+x}{\sqrt{2}+\sqrt{(2+x)}} + \dfrac{2-x}{\sqrt{2}-\sqrt{(2-x)}} = \sqrt{2}$.

3. $a + x = \sqrt{\{a^2 + x\sqrt{(b^2+x^2)}\}}$.

4. $\dfrac{\sqrt{(a^2+x^2)}+x}{\sqrt{(a^2+x^2)}-x} = \dfrac{b}{c}$. 5. $\dfrac{ax-b^2}{\sqrt{(ax)}+b} = \dfrac{\sqrt{(ax)}-b}{n} - c$.

6. $\dfrac{2}{x+\sqrt{(2-x^2)}} + \dfrac{2}{x-\sqrt{(2-x^2)}} = x$.

7. $\dfrac{\sqrt{(2+x)}+\sqrt{(2-x)}}{\sqrt{(2+x)}-\sqrt{(2-x)}} = 2$. 8. $\dfrac{ax-1}{\sqrt{(ax)}+1} = 4 + \dfrac{\sqrt{(ax)}-1}{2}$.

9. $\dfrac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}-\sqrt{(a-x)}} = a$.

10. $\dfrac{\sqrt{(x-a+b)}-\sqrt{(x+a-b)}}{\sqrt{(x-a+b)}+\sqrt{(x+a-b)}} = \dfrac{a-b}{a+b}$.

11. $\dfrac{1-\sqrt{\{1-\sqrt{(1-x)}\}}}{1+\sqrt{\{1-\sqrt{(1-x)}\}}} = b$.

12. $\dfrac{1-ax}{1+ax} \cdot \sqrt{\left(\dfrac{1+bx}{1-bx}\right)} = 1$.

13. $\dfrac{2.7x-1}{\sqrt{(2.7x)}+1} = \dfrac{\sqrt{(2.7x)}+7}{2}$.

14. $\dfrac{\sqrt{(a^2-x^2)}-\sqrt{(b^2+x^2)}}{\sqrt{(a^2-x^2)}+\sqrt{(b^2+x^2)}} = \dfrac{c}{d}$.

15. $\sqrt[3]{(1+x)^2} + \sqrt[3]{(1-x^2)} + 2\sqrt[3]{(1-x)^2} = 0$.

16. $\sqrt[3]{(a+x)} + \sqrt[3]{(a-x)} = c.$

17. $\sqrt[3]{(1+2x)} + \sqrt[3]{(1-2x)} = \sqrt[3]{c}.$

18. $(1+x)^{\frac{2}{5}} + \frac{3}{16}(1-x)^{\frac{2}{5}} = (1-x^2)^{\frac{1}{5}}.$

201. Square Root of Polynomials.—In practice we can usually find by inspection the *square root of a polynomial.* But a *general rule* may be derived from Formula A, for the *square root of a binomial.*

Ex. 1.

$$\begin{array}{rl}
 & a^2 + 2ab + b^2 \;\big|\; \underline{a+b} \\
 & \underline{a^2 \qquad\qquad} \\
\underline{2a+b} \,\big| & +2ab + b^2 = (2a+b) \times b \\
 & \underline{+2ab + b^2}.
\end{array}$$

The square of the binomial is made up of 1°. *the square of the first term;* 2°. *Twice* the *first* term into the *second*, and 3°. the *square* of the *second* term. Hence,

1°. We arrange in descending (or ascending) powers of a.

2°. We find the square root, a, of the first term, and subtract its square from the given expression.

3°. We divide the first term of the remainder by *twice* a, getting $+b$, the second term.

4°. We add $+b$ to $2a$, and multiply the sum by b.

The student should observe that a and b *stand for* any *number of terms* (Art. 82); and that a stands for the *first part* of the required root, and b for the *next term. Twice* a is always the *trial divisor* for the successive terms of the root.

E. g. $(a+b+c)^2 = \{(a+b)+c\}^2 = (a+b)^2 + 2(a+b)c + c^2$

$$= a^2 + (2a+b)b + \{2(a+b)+c\}c$$

$(a+b+c+d)^2 = \{(a+b+c)+d\}^2 = (a+b+c)^2 + 2(a+b+c)d + d^2$

$$= a^2 + (2a+b)b + \{2(a+b)+c\}c + \{2(a+b+c)+d\}d, \text{ etc., etc.;}$$

where the *law* shown in Ex. 1 plainly holds.

Ex. 2.

$$\begin{array}{rl}
 & 9x^6 - 24x^4y^2 - 12x^3y^3 + 16x^2y^4 + 16xy^5 + 4y^6 \;\big|\; \underline{3x^3 - 4xy^2 - 2y^3} \\
 & 9x^6 \\
\underline{6x^3 - 4xy^2} \,\big| & -24x^4y^2 - 12x^3y^3 + 16x^2y^4 \\
 & \underline{-24x^4y^2 \qquad\qquad + 16x^2y^4} \\
\underline{6x^3 - 8xy^2 - 2y^3} & -12x^3y^3 + 16xy^5 + 4y^6 \\
 & \underline{-12x^3y^3 + 16xy^5 + 4y^6}.
\end{array}$$

202. Cube Root of Polynomials.—Similarly, a *general rule* for extracting the *cube root* of a quantity, may be derived from Formula G. (Art. 91) for the *cube* of a *binomial.*

Ex. 1.

$$\begin{array}{r|l} & a^3+3a^2b+3ab^2+b^3 \;|\; \underline{a+b} \\ & \underline{a^3} \\ 3a^2+3ab+b^2 & +3a^2b+3ab^2+b^3=(3a^2+3ab+b^2)\times b \\ \underline{b} & \\ & \underline{=+3a^2b+3ab^2+b^3.} \end{array}$$

1°. Arrange in descending powers of a.

2°. Extract the cube root (a) of the *first* term, for the *first term* of the root, and subtract its cube from the given expression.

3°. Divide the *first* term of the remainder by 3 *times the square* of a, for the *second term* (b) of the root.

4°. Add to the *trial-divisor* ($3a^2$), b^2 and 3 times the product of a and b, and multiply the whole by b; this gives the complete subtrahend. If there are more than two terms in the root, the process is the same; thus, $a+b+c$, is $(a+b)+c$, where $a+b$ is the *first* term, etc.

$$\begin{aligned} E.\ g.\ (a+b+c)^3 &= \{(a+b)+c\}^3 \\ &= (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3, \\ &= a^3+(3a^2+3ab+b^2)b+\{3(a+b)^2+3(a+b)c+c^2\}c. \\ (a+b+c+d)^3 &= (a+b+c)^3+3(a+b+c)^2d+3(a+b+c)d^2+d^3 \\ &= a^3+(3a^2+3ab+b^2)b+\{3(a+b)^2+3(a+b)c+c^2\}c \\ &\quad +\{3(a+b+c)^2+3(a+b+c)d+d^2\}d, \text{ etc., etc.,} \end{aligned}$$

where the law exemplified in Ex. 1., plainly operates.

Ex. 2.

$$\begin{array}{rcl} & & x^6+6x^5+15x^4+20x^3+15x^2+6x+1 \;|\; \underline{x^2+2x+1} \\ & & \underline{x^6} \\ 3\cdot(x^2)^2+3x^2\cdot 2x+4x^2 & & +6x^5+15x^4+20x^3 \\ \underline{2x} & & \\ \ldots\ldots\ldots & = & \ldots\ \underline{6x^5+12x^4+\ 8x^3} \\ 3(x^2+2x)^2+3(x^2+2x)\cdot 1+1^2 & & +\ 3x^4+12x^3+15x^2+6x+1 \\ \underline{1} & & \\ \ldots\ldots\ldots\ldots & = & \ldots\ \underline{+\ 3x^4+12x^3+15x^2+6x+1} \end{array}$$

N. B.—If a *fourth* root is required, we take the square root of the square root, *i. e.* $(a^{\frac{1}{2}})^{\frac{1}{2}}=a^{\frac{1}{4}}$; etc. So a *cube* root of a *square* root gives a sixth root, *i. e.* $(a^{\frac{1}{2}})^{\frac{1}{3}}=a^{\frac{1}{6}}$; etc.

203. Complete Squares and Cubes.—We give a few examples of questions that may be solved by the methods explained in the last two articles.

Ex. 1. Find the condition that ax^2+bx+c may be a perfect square. Applying the method of extracting square root, we get the root $a^{\frac{1}{2}}x+b\div 2a^{\frac{1}{2}}$, and remainder $c-\frac{b^2}{4a}$; now if the quantity is to be a perfect square this remainder must vanish, *i. e.*, $c-\frac{b^2}{4a}=0$, or $b^2-4ac=0$, the required condition.

Ex. 2 What value of x will make $x^4-2ax^3+(a^2+2b)x^2-3abx+2b^2$, a complete square? Proceeding as in last example; the root is found to be x^2-ax+b, and the remainder is $-abx+b^2$, which must vanish; *i. e.*, $-abx+b^2=0$; $\therefore\ x=\frac{b}{a}$.

Ex. 3. Find the relation between b and c in order that x^3+3ax^2+bx+c may be a complete cube for all values of x. Extracting the cube root, we get $x+a$, with remainder $(b-3a^2)x+c-a^3$, which must vanish for all values of x;

$$\therefore\quad b-3a^2=0,\quad \text{or}\quad a^6=b^3\div 27,$$

and

$$c-a^3=0,\quad \text{or}\quad a^6=c^2,$$

$$\therefore\quad c^2=b^3\div 27,\quad \text{or}\quad b^3=27c^2,$$

the required relation.

N. B.—Ex. 3 and similar ones may be treated thus:—We see that the first term of the cube root is x, and the second term must be a, therefore assume

$$x^3+3ax^2+bx+c=(x+a)^3=x^3+3x^2a+3xa^2+a^3;$$

which is true for all values of x, and, therefore, the **coefficients of the like powers must be equal,** *i. e.*,

$$b=3a^2,\quad \text{or}\quad a^6=b^3\div 27;$$

$$c=a^3,\quad \text{or}\quad a^6=c^2,\ \text{as before.}$$

EXERCISE XC~~VI.~~ IV

(*a*)

Find the square roots of :—

1. $16x^6-24x^5+25x^4-20x^3+10x^2-4x+1.$ $4x^3-3x^2+2x-$
2. $64x^6-192x^5+240x^4-160x^3+60x^2-12x+1.$

3. $64x^6 - 192x^5y + 240x^4y^2 - 160x^3y^3 + 60x^2y^4 - 12xy^5 + y^6$.

4. $x^6 - 4x^5y + 8x^4y^2 - 10x^3y^3 + 8x^2y^4 - 4xy^5 + y^6$.

5. $4 - 12x + 5x^2 + 14x^3 - 11x^4 - 4x^5 + 4x^6$.

6. $x^8 - 4x^7y + 4x^6y^2 + 6x^5y^3 - 14x^4y^4 + 4x^3y^5 + 9x^2y^6 - 6xy^7 + y^8$.

7. Find $\sqrt{(1 + a)}$ to five terms. $x^4 - 2x^3y + 3xy^3 - y^4$

(*b*)

Find the cube roots of :—

1. $x^6 - 3x^5 + 6x^4 - 7x^3 + 6x^2 - 3x + 1$. $x^2 - x + 1$

2. $8 - 48x + 108x^2 - 112x^3 + 54x^4 - 12x^5 + x^6$. $2 - 4x + x^2$

3. $1 + 9x + 24x^2 + 9x^3 - 24x^4 + 9x^5 - x^6$. $1 + 3x - x^2$

4. $y^6 - 3y^5 + 9y^4 - 13y^3 + 18y^2 - 12y + 8$. $y^2 - y + 2$

5. $x^9 - 3x^8y + 6x^7y^2 - 4x^6y^3 + 6x^4y^5 - 2x^3y^6 + 3xy^8 + y^9$.
$x^3 - x^2y + 3xy^2 + y^3$

6. $a^6 - a^3 + a^2 + \frac{1}{3} - \frac{2}{3}a^{-1} + \frac{1}{3}a^{-2} - \frac{1}{27}a^{-3} + \frac{1}{9}a^{-4} - \frac{1}{9}a^{-5} + \frac{1}{27}a^{-6}$.
$a^2 - \frac{1}{3}a^{-1} + \frac{1}{3}a^{-2}$

7. Find $\sqrt[3]{(1 + x)}$ to four terms.
$1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3$

(*c*)

Find the values of x which will make the following perfect squares :—

1. $x^4 - 4x^3 + 6x^2 - 8x + 8$.

2. $x^4 - 2x^3 - x^2 + 4x - 2$.

3. $x^4 + 6x^3 + 11x^2 + 3x + 31$.

4. $9x^4 + 12x^3 + 10x^2 - 8x - 8$.

5. $x^4 + 2ax^3 + 3a^2x^2 + cx + d$.

6. $x^4 - 2ax^3 + (a^2 + 2b)x^2 - 4abx + 3b^2$.

Find values of x which will make the following perfect cubes :—

7. $x^3 + 6x^2 + 14x - 8$.

8. $8x^3 - 36x^2 + 56x - 39$.

9. $x^3 + 12x^2 + 60x + 70$.

10. $\frac{1}{27}x^6 - \frac{1}{3}a^2x^4 + 4a^4x^2 - 28a^6$.

11. Find the condition that $(2m - x)(2n - x) - a^2$ may be a perfect square.

12. If $p^2x^2 + qx + qr + q^2$ be a perfect square, show that

$$\frac{1}{4p^2} - \frac{r}{q} = 1.$$

13. If $a^3x^6 - 9a^2bx^4 + 39ab^2x^2 - 51b^3$ be a perfect cube, then

$$x = \pm\sqrt{\left(\frac{2b}{a}\right)}.$$

14. If $ax^3 + bx^2 + cx + d$ be a complete cube, then will

$$ac^3 = db^3, \quad \text{and} \quad b^2 = 3ac.$$

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